Absolutely Continuous Invariant Measures for Nonuniformly Expanding Maps

Huyi Hu, Sandro Vaienti

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Abstract

For a large class of nonuniformly expanding maps of $\mathbb{R}^m$, with indifferent fixed points and unbounded distorsion and non necessarily Markovian, we construct an absolutely continuous invariant measure. We extend to our case techniques previously used for expanding maps on quasi-Hölder spaces. We give general conditions and provide examples to which apply our result.

0 Introduction

A challenge problem in smooth ergodic theory is to construct invariant measures for multidimensional maps T with some sort of weak hyperbolicity and then to study their statistical properties (decay of correlations, central limit theorem, distribution of return times, etc.). For nonuniformly expanding endomorphisms of $\mathbb{R}^m$, only few results exist at the moment. When the system has a Bernoulli structure, or verifies the so-called "finite range structure", and it enjoys a suitable distortion relation (Renyi's condition), M. Yuri [22, 23] was able to construct an invariant, possibly σ-finite, measure absolutely continuous with respect to the Lebesgue measure. Young's tower ([20, 21]), which is mainly for nonuniformly hyperbolic systems, also works for nonuniformly expanding maps, and the invariant measures and other statistical properties can be obtained, if some bounded distortion properties are assumed. In Alves-Bonatti-Viana's work ([2], also see [3, 4, 5]), nonuniformly expansion are understood as the average value of $\log ||DT(x)^{-1}||$ along the orbits to be less than zero for almost every points. Under some conditions on the set of critical points, they can

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construct an absolutely continuous invariant measure. Recently this theory has been applied to maps which allow contraction in some regions [15, 6].

The aim of our paper is to treat a class of nonsingular transformations with indifferent fixed points which do not enjoy any Markov property. We obtain existence of absolutely continuous invariant measures that can be finite or infinite, depending on the behaviour of T near the fixed point. The technique we use consists of the following steps. We first replace the transformation with the first return map with respect to the domain outside a small region around the indifferent fixed point. What we get is a uniformly expanding map with a countable number of discontinuity surfaces. Then we prove a Lasota-Yorke [14] inequality on the induced space by acting the Perron-Frobenius operator on the space of "quasi-Hölder" functions, particularly adapted when the invariant densities are discontinuous. As soon as the Lasota-Yorke inequality has been proved, simple compactness argument will allow us to apply the Ionescu-Tulcea and Marinescu theorem to conclude that there exists an absolutely continuous invariant measure. The space of quasi-Hölder functions, introduced by Keller [13], developed by Blank [7] and successfully applied by Saussol [17] and successively by Buzzi [9] (see also [10]) and Tsujii [19] to the multidimensional expanding case, reveals to be very useful to control the oscillations of a function under the iteration of the PF operator across the discontinuities of the map. The use of the more standard space of bounded variation functions allowed as well to get absolute invariant measures for a wide class of piecewise expanding maps, see, for instance [8, 16, 1, 11].

In adapting to our situation the Saussol's strategy to prove the Lasota-Yorke inequality, the difficult part comes from the indifferent fixed points. Unlike in one dimensional case, the maps in higher dimensional space have unbounded distortion *away from* the indifferent fixed points, that is, there are uncountably many points x, whose neighborhoods contain points y, arbitrary close to x, such that the distortion of $|\det DT|$ is unbounded along the backward orbits towards the indifferent fixed point (see Example 1 in Section 2). This forced us to a certain number of assumptions which basically reduce to insure sufficiently good expanding rates in a small neighborhood of the neutral point, and insure bounded distortion along the curves close to radial directions (Assumption 4(b) and (c)). A careful view at the proofs will reveal that such assumptions are unavoidable, unless to modify deeply all the structure of the approach. We nevertheless point out that our hypothesis could be easily verified on some simple cases if the local behavior of the map T near the indifferent fixed points is understood. On the other hand, it seems that other known techniques are difficult to apply. Since we know that distortions are unbounded for the maps we are interested in, Young's results cannot be applied directly. Also, the condition $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||DT(T^i(x))^{-1}|| < 0$ in [2] cannot be obtained in our case (and in fact it fails if T admit a σ-finite absolutely continuous invariant measures). If we study the first return maps $\hat{T}$ instead, then $\|D\hat{T}_x(v)\|$ can be arbitrary large for x close to the discontinuity set, and therefore the assumptions on the critical set in [2] are not satisfied.

The plan of the paper is the following: in Section 1 we state the assumptions and the main theorems, A and B. Section 2 is devoted to examples. The proofs of the main results are in Section 3 through 6.

1 Assumptions and statements of results

Let $M \subset \mathbb{R}^m$ be a compact subset with $\overline{\operatorname{int} M} = M$ and d be the Euclidean distance. Let ν be the Lebesgue measure on M. We assume $\nu M = 1$.

For $A \subset M$ and $\varepsilon > 0$, denote $B_\varepsilon(A) = \{x \in \mathbb{R}^m; d(x, A) \le \varepsilon\}$.

Let $T : M \to M$ be an almost expanding piecewise smooth map with an indifferent fixed point p.

We assume that T satisfies the following assumptions.

Assumption 1. (Piecewise smoothness) *There are finitely many disjoint open sets $U_1, \cdots, U_K$ with $M = \bigcup_{i=1}^K \overline{U}_i$ such that for each i,*

(a) *$T_i := T|_{U_i} : U_i \to M$ is $C^{1+\alpha}$;*

(b) *T_i can be extended to a $C^{1+\alpha}$ map $T_i : \tilde{U}_i \to M$ such that $T_i\tilde{U}_i \supset B_{\varepsilon_1}(T_iU_i)$ for some $\varepsilon_1 > 0$, where $\tilde{U}_i$ is a neighborhood of U_i.*

Assumption 2. (Fixed point) *There is a point $p \in U_1$ such that:*

(a) *$Tp = p$;*

(b) *$T^{-1}p \notin \partial U_j$ for any j.*

Since $M \subset \mathbb{R}^m$, we may take a coordinate system such that $p = 0$. Hence, we write $|x| = d(x, p)$ if $x \in M$.

For any $x \in U_i$, we define $s(x) = s(x, T)$ by

$$s(x, T) = \min\big\{s : d(x, y) \le s d(Tx, Ty), y \in U_i, d(x, y) \le \min\{\varepsilon_1, 0.1|x|\}\ \big\}.$$

Denote by γ_m the volume of the unit ball in $\mathbb{R}^m$.

Assumption 3. (Expanding Rates) *There exists an open region R bounded by a smooth surface with $p \in R$, $\overline{R} \subset TR$, $\overline{TR} \subset U_1$ and with either $\overline{R} \subset TU_j$ or $\overline{R} \cap TU_j = \emptyset$ such that:*

(a) *$0 < s(x) \le 1 \quad \forall x \in M \setminus \{p\}$, and if $s(x) = 1$ then $x \in R$ and $|Tx| > |x|$;*

(b) *there exist constants $\eta_0 \in (0, 1)$, $\varepsilon_2 > 0$ such that*

$$s^\alpha + \lambda \le \eta_0 < 1,$$

where

$$s := \max\{s(x) : x \in M\backslash R\},$$

$$\lambda = \max\Big\{2 \sup_{\varepsilon_0 \le \varepsilon_2} \sup_{\varepsilon \le \varepsilon_0} \frac{G_U(\varepsilon, \varepsilon_0)}{\varepsilon^\alpha}\varepsilon_0^\alpha,\ \frac{3s\gamma_{m-1}}{(1-s)\gamma_m}\Big\}, \tag{1.1}$$

$$G_U(\varepsilon, \varepsilon_0) = \sup_{x\in M} G_U(x, \varepsilon, \varepsilon_0), \tag{1.2}$$

and

$$G_U(x, \varepsilon, \varepsilon_0) = \sum_{j=1}^{K} \frac{\nu(T_j^{-1}B_\varepsilon(\partial TU_j) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))};$$

(c) *there exists $N = N_s > 0$ and $\varepsilon_3 > 0$ such that for all $x \in R_{\varepsilon_3}(TR\backslash R)$,*

$$s(T_1^{-N}(x), T_1^N) \le \frac{s}{5m}\Big(\frac{\lambda(1-s)^m}{2C_\xi I^2}\Big)^{1/\alpha}$$

for λ given by (1.1) and I and C_ξ given by Assumption 4(c).

Remark 1.1. *By Assumption 3(a), the map $T_j : U_j \to T_j(U_j)$ is noncontracting for each j, and therefore it is a local diffeomorphisms. Also, by the assumption, for any $x \in U_1$, $T_1^{-n}x \to p$, because the set of limit points of $\{T_1^{-n}x\}$ cannot contain any other point but p.*

Remark 1.2. *Assumption 3(b) is the main assumption that requires uniformly expanding outside R and gives condition on the relations between expanding rates and discontinuity. We refer to [17] for more details about the meaning of $G_U(\varepsilon, \varepsilon_0)$. (In fact, for small ε_0, $G_U(\varepsilon, \varepsilon_0)\varepsilon_0/\varepsilon$ is greater than $4s\gamma_{m-1}/(1-s)\gamma_m^{-1}$ if there are at least two surfaces ∂U_i meet at some point. See Lemma 2.1 in [17].)*

Remark 1.3. *Assumption 3(b) implies $\nu(\partial U_j) = 0$ for any $j = 1, \cdots, K$.* [*]

[*] In fact, if $\nu(\partial U_j) > 0$ for some j, then we take the set of the *density points*

$$\Delta = \left\{ x \in M : \lim_{\varepsilon\to 0} \frac{\nu(B_\varepsilon(x) \cap \partial U_j)}{\nu B_\varepsilon(x)} = 1 \right\}.$$

By the Lebesque-Vitali Theorem (see, e.g. [18], Chapter 10), $\nu\Delta = \nu(\partial U_j) > 0$. In particular, $\Delta \ne \emptyset$. Therefore for any $x \in \Delta$, if ε_0 is sufficiently small and $\varepsilon = (1-s)\varepsilon_0$, then

$$G_U(x, \varepsilon, \varepsilon_0) \ge \frac{\nu(T_j^{-1}B_\varepsilon(\partial TU_j) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))} \ge \frac{\nu(\partial U_j \cap B_\varepsilon(x))}{\nu(B_\varepsilon(x))}$$

is sufficiently close to 1, which contradicts to the assumtion.

Remark 1.4. *We allow that $s(x,T)=1$ for some x other than p. However we still need some expanding rate inside R. This is given by Assumption 3(c). If $s(T_1^{-N}(x),T_1^N)$ can be arbitrarily small by taking N sufficiently large, then Assumption 3(c) is always true.*

Denote $R_0=TR\backslash R$. Clearly, $R_0\subset U_1$ because of the choice of R.

Assumption 4. (Distortions)

(a) *There exists $c>0$ such that for any $x,y\in TU_j$ with $d(x,y)\le\varepsilon_1$,*

$$\left|\det DT_j^{-1}(x)-\det DT_j^{-1}(y)\right|\le c|\det DT_j^{-1}(x)|d(x,y)^\alpha,$$

where ε_1 is given by Assumption 1(b);

(b) *For any $b>0$, there exist $J>0$, $\varepsilon_4>0$ such that for any $\varepsilon\in(0,\varepsilon_4]$, we can find $0<N=N(\varepsilon)\le\infty$ with*

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|}\le 1+J\varepsilon^\alpha\quad\forall y\in B_\varepsilon(x),\ x\in B_{\varepsilon_4}(R_0),\ n\in(0,N],$$

and

$$\sum_{n=N}^{\infty}\sup_{y\in B_\varepsilon(x)}|\det DT_1^{-n}(y)|\le b\varepsilon^{m+\alpha}\quad\forall x\in B_{\varepsilon_4}(R_0);$$

(c) *There exist constants $I>1$, $C_\xi>0$, $\varepsilon_5>0$ such that for any $0<\varepsilon_0\le\varepsilon_5$, $n>0$, there is a finite or countable partition $\xi=\xi_n$ of $B_{\varepsilon_0}(R_0)$ such that $\forall A\in\xi$, $0<\varepsilon\le\varepsilon_0$, $\mathrm{diam}(A\cap B_{\varepsilon_0}(\partial R_0))\le 5m\varepsilon_0$,*

$$\frac{\nu\big(B_\varepsilon(\partial R_0)\cap A)\big)}{\nu\big(B_{\varepsilon_0}(\partial R_0)\cap A\big)}\le C_\xi\left(\frac{\varepsilon}{\varepsilon_0}\right)^\alpha, \tag{1.3}$$

whenever $\nu\big(T_1^{-n}(B_{\varepsilon_0}(\partial R_0))\cap A\big)\ne 0$, and for any $x,y\in A$,

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|}\le I. \tag{1.4}$$

Remark 1.5. *In fact, Assumption 4(a) is a consequence of Assumption 1.* † *However, we state it here independently due to its importance for our arguments.*

Remark 1.6. *If T_1^{-1} has bounded distortion in $B_{\varepsilon_5}(R_0)$ in the sense that for any $J_0>1$, there is $\varepsilon>0$ such that for any $x,y\in B_{\varepsilon_5}(R_0)$ with $d(x,y)\le\varepsilon$ and for any $n>0$, $\dfrac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|}\le J_0 d(x,y)^\alpha$, then Assumption 4(b) and (c) are true with $\varepsilon_4=\varepsilon_5=\varepsilon_0$.*

† we note that by Assumption 1(b), the map $x\to|\det DT(x)|$ is continuous on $\overline{U}_i$ for each i. Since $\overline{U}_i$ is compact, $|\det DT(x)|$ is bounded. Hence, Assumption 4(a) follows from the fact that T is piecewise $C^{1+\alpha}$, Assumption 1(a)

Remark 1.7. *Actually, by our proof the condition* $\operatorname{diam}(A \cap B_{\varepsilon_0}(\partial R_0)) \le 5m\varepsilon_0$ *in Assumption 4(c) can be replace by*

$$\operatorname{diam} T_1^{-n}(A \cap B_{\varepsilon_0}(\partial R_0)) \le s\Big(\frac{\lambda(1-s)^m}{2C_\xi I^2}\Big)^{1/\alpha}$$

for all $n \ge N_s$*, where* s *and* N_s *are given by Assumption 3(b) and (c) respectively (see (6.4)).*

Remark 1.8. *When we iterate the system, oscillations of the test functions are produced by both discontinuities* ∂U_j *and distortion of* $|\det DT|$*. It is very common for an expanding system in multidimensional space with an indifferent fixed point to have unbounded distortion near the fixed point. (See Example 1 in Section 2). Assumption 4(b) requires that either the distortion of* $|\det DT_1^{-n}(x)|$ *or* $|\det DT_1^{-n}(x)|$ *itself is small. On the other hand, if the distortion of* $|\det DT_1^{-n}(x)|$ *is bounded along the radial direction, then Assumption 4(c) holds.*

Theorem A. *Suppose* $T : M \to M$ *satisfies Assumption 1-4. Then* T *admits an absolutely continuous invariant measure* μ *with at most finitely many ergodic components* $\mu_1, \cdots, \mu_s$ *that are either finite or* σ*-finite, and the density functions of* μ_i *are bounded on any compact set away from* p*. Hence,*

- μ *is finite if* $\displaystyle\sum_{n=1}^{\infty} \nu(T_1^{-n}R) < \infty$.

Moreover, if $|\det DT|$ *is bounded and for any ball* $B_\varepsilon(x)$ *in* M*, there exists* $\tilde N = \tilde N(x,\varepsilon) > 0$ *such that* $T^{\tilde N} B_\varepsilon(x) \supset M$*, then the density function is bounded below by a positive number. Hence*

- μ *is* σ*-finite if* $\displaystyle\sum_{n=1}^{\infty} \nu(T_1^{-n}R) = \infty$.

Remark 1.9. *We will give an example in Section 2 showing that it is possible for* μ *to have both finite and* σ*-finite ergodic components simultaneously, and both contain the same indifferent fixed point* p *in their supports.*

Since Assumption 4(b) and 4(c) are difficult to verify, we give some sufficient conditions in the next theorem.

One of the interesting cases we would discuss is the following: there are constants $\gamma' > \gamma > 0$, $C_i, C_i' > 0$, $i = 0,1,2$, such that

$$|x|\big(1 - C_0'|x|^\gamma + O(|x|^{\gamma'})\big) \le |T_1^{-1}x| \le |x|\big(1 - C_0|x|^\gamma + O(|x|^{\gamma'})\big), \quad (1.5)$$

$$1 - C_1'|x|^\gamma \le \|DT_1^{-1}(x)\| \le 1 - C_1|x|^\gamma, \quad (1.6)$$

$$C_2'|x|^{\gamma-1} \le \|D^2T_1^{-1}(x)\| \le C_2|x|^{\gamma-1}. \quad (1.7)$$

If T satisfies all of the inequalities, then $\|DT_p\| = 1$. So DT_p is either the identity or a rotation. If T satisfies the second inequalities in (1.5)-(1.7), then $\|DT_p\|$ may have eigenvalues greater than 1.

In the theorem below, we denote by $E(v_1, \cdots, v_k)$ the subspace spanned by vectors $v_1, \cdots, v_k$, and by $E_x(S)$ the tangent space of a submanifold S at a point $x \in S$. Also, we may use a coordinate system (t, ϕ) near p where $t = |x|$ and $\phi \in \mathbb{S}^{m-1}$, the $m-1$ dimensional sphere.

Theorem B. *Suppose $T : M \to M$ satisfies Assumption 1-3 and 4(a). Assumption 4(b) and 4(c) are satisfied if the conditions in Part (I) and (II) below hold respectively. Hence, the conclusions of Theorem A hold.*

I) One of the following conditions holds:

i) There exists a constant $\kappa \in (0,1)$ such that $|\det DT| \geq \kappa^{-1} > 1$, and a constant $\hat{\alpha} > \alpha$ such that T is $C^{1+\hat{\alpha}}$ in a neighborhood of p. In this case, μ is finite if Assumption 4(c) also holds.

ii) There exists an open region $\tilde{R} \subset R$ containing p with $T_1^{-L}R \subset \tilde{R}$ for some $L > 0$, and constants $\gamma' > \gamma > 0$, $C_0, C_1, C_2 > 0$ such that the second inequalities in (1.5)-(1.7) hold; and there exist constants $\delta, \tau > 0$, $C_\delta, C_\tau > 0$ with

$$\frac{1}{\gamma(1-\alpha)} - \tau < \frac{\delta - 1}{m + \alpha} \tag{1.8}$$

such that for any $x \in R_0$, $n \geq L$,

$$|\det DT_1^{-n}(x)| \leq \frac{C_\delta}{n^\delta}, \qquad \|DT_1^{-n}(x)\| \leq \frac{C_\tau}{n^\tau}. \tag{1.9}$$

II) One of the following conditions holds:

i) There is a decomposition of TR into finite or countable number of cones $\{\mathcal{C}_i\}$ and a partial order "$\prec$" on each $\mathcal{C}_i \cap R$ such that $\nu(R \setminus \cup_i \mathcal{C}_i) = 0$ and $T(\mathcal{C}_i \cap R) = \mathcal{C}_i \cap TR$; $x \prec Tx$ for any $x \in \mathcal{C}_i \cap R$ and for any $y \in R_0$ there is $x \in \partial R$ such that $x \prec y \prec Tx$; $x \prec y$ implies $T_1^{-1}x \prec T_1^{-1}y$ and $|\det DT(x)| \leq |\det DT(y)|$.

ii) Suppose T is $C^{1+\gamma}$ and satisfies (1.5)-(1.7) near p. There are two families of cones $\{\mathcal{C}_x\}$ and $\{\mathcal{C}'_x\}$, continuous uniformly in (t, ϕ), where $t \geq 0$ and $\phi = \mathbb{S}^{m-1}$ with $(t, \phi) \in TR$, in the tangent bundle over the set TR such that (a) $DT_x(\mathcal{C}_x) \subset \mathcal{C}_{Tx}$ and $DT_x(\mathcal{C}'_x) \supset \mathcal{C}'_{Tx}$ $\forall x \in R$; (b) there exists a positive angle θ_0 such that for any $x \in TR$ and $v \in \mathcal{C}_x$ and $v' \in \mathcal{C}'{}_x$, the angle between these two vectors is bounded from below by θ_0; (c) $\exists d > 0$, such that

$$\frac{|\det DT_x|_{E(v,v')}|}{\|DT_x|_{E(v)}\| \cdot \|DT_x|_{E(v')}\|} \leq 1 - d|x|^\gamma \tag{1.10}$$

for any $v, v' \in \mathcal{C}_x$; and (d) $\mathcal{C}_x$ contains the position vector from p to x for all $x \in TR$, $\mathcal{C}'_x$ contains $E_x(\partial B_\varepsilon(R_0))$ for all $x \in \partial(B_\varepsilon(R_0))$, $0 < \varepsilon \le \varepsilon_5$, and

$$\|DT_x|_{E(\partial(T_1^{-n}R))}\| \le \frac{|Tx|^{1/(1-\theta)}}{|x|^{1/(1-\theta)}} \quad \forall x \in \partial(T_1^{-n}R),\ n > 0 \quad (1.11)$$

for some θ with $(1+\gamma)(1-\theta) > 1$.

Remark 1.10. *The condition in Theorem B.I).i) means that DT_p has at least one eigenvalue with absolute value greater than 1.*

The condition in Theorem B.II).ii) part (c) implies that under DT, vectors in the cone $\mathcal{C}_x$ expands faster than that in $\mathcal{C}'_x$.

Remark 1.11. *If we write $DT(x) = T_0(x) + T_\gamma(x) + T_h(x)$, where $T_0 = DT_p$, T_γ satisfies $T_\gamma(tx) = t^\gamma T_\gamma(x)$ $\forall t > 0$ and $|T_h(x)| = O(|x|^{\gamma'})$, $\gamma' > \gamma$, then the cones $\{\mathcal{C}_x\}$ and $\{\mathcal{C}'_x\}$ are mainly determined by T_γ as x near p. So it is easy to get uniformity near $t = 0$.*

2 Examples

In the next example we show that near an indifferent fixed point p of a map $T : \mathbb{R}^m \to \mathbb{R}^m$, distortion may be unbounded even away from p in the sense that there is a point z such that for any neighborhood V of z, we can find $\hat{z} \in V$ such that the ratio

$$|\det DT_1^{-n}(z)|/|\det DT_1^{-n}(\hat{z})| \quad (2.1)$$

is unbounded as $n \to \infty$.

Example 1. *Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ in such a way around $(0,0)$ it behaves like:*

$$T(x,y) = \big(x(1+x^2+y^2),\ y(1+x^2+y^2)^2\big). \quad (2.2)$$

It is easy to see that

$$DT(x,y) = \begin{pmatrix} 1+3x^2+y^2+O(|z|^4) & 2xy+O(|z|^4) \\ 4xy+O(|z|^4) & 1+2x^2+6y^2+O(|z|^4) \end{pmatrix}, \quad (2.3)$$

and

$$\det DT(x,y) = 1+5x^2+7y^2+O(|z|^4), \quad (2.4)$$

where $z = (x,y)$ and $|z| = \sqrt{x^2+y^2}$.

Note that in this example, T is locally injective and T^{-1} will denote its inverse. Take $z' = (x_0, 0)$ and denote $z'_n = T^{-n}z'$. By Lemma 3.1 in the next section, we have $|z'_n| \sim \dfrac{1}{\sqrt{2n}}$, where $a_n \sim b_n$ means $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$. Hence by

(2.4) and Lemma 3.2, $|\det DT^{-n}(z')| \leq \dfrac{D'}{n^{5/2}}$ for some $D' > 0$. On the other hand if we take $z'' = (0, y_0)$ and denote $z''_n = T^{-n}z''$, then $|z''_n| \sim \dfrac{1}{\sqrt{4n}}$ and $|\det DT^{-n}(z'')| \geq \dfrac{D''}{n^{7/4}}$ for some $D'' > 0$. So $\dfrac{|\det DT^{-n}(z'')|}{|\det DT^{-n}(z')|} \to \infty$ as $n \to \infty$.

Suppose that for every $z \neq (0,0)$, there is a neighborhood V such that for all $\hat{z} \in V$, the ratio in (2.1) is bounded for all $n > 0$. We take a curve from z' to z'' that does not contain the origin. By choosing finite cover on the curve, we know that the ratio $|\det DT^{-n}(z'')|/|\det DT^{-n}(z')|$ should be bounded. This is a contradiction. It means that there are some points away from $(0,0)$ at which distortion is unbounded.

In the next two examples we show how to get Assumption 4(b) and 4(c) by applying Theorem B.

Example 2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by*

$$T(x,y,z) = \big(x(1 + x^2 + y^2 + z^2), y(1 + x^2 + y^2 + z^2)^2, z(2 + x^2 + y^2 + z^2)^3\big)$$

as (x,y,z) near the origin.

Note that by similar arguments as above we know that for this map the distortion is also unbounded away from the origin.

Since $\det DT_{(0,0,0)} = 2$ and T is C^∞ near the origin, by Theorem B.I).i), Assumption 4(b) is satisfied.

Let $\mathcal{C}_i$, $i = 1, \cdots, 8$, be the eight octants in $\mathbb{R}^3$, and define a partial order "$\prec$" by letting $w_1 = (x_1, y_1, z_1) \prec w_2 = (x_2, y_2, z_2)$ if $|x_1| \leq |x_2|$, $|y_1| \leq |y_2|$ and $|z_1| \leq |z_2|$. Clearly all the requirements in Theorem B.II).i) are satisfied. So we get Assumption 4(c) as well.

Example 3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as in the first example.*

For any $z = (x, y)$, we denote $z_n = T^{-n}z$.

Note that

$$|z|(1 + |z|^2 + O(|z|^4)) \leq |Tz| \leq |z|(1 + 2|z|^2 + O(|z|^4)),$$

or

$$|z_n|\big(1 + |z_n|^2 + O(|z_n|^4)\big) \leq |z_{n-1}| \leq |z_n|\big(1 + 2|z_n|^2 + O(|z_n|^4)\big).$$

So by Lemma 3.1, we have

$$\frac{1}{\sqrt{4(n+k)}} + O(n^{-\beta'}) \leq |z_n| \leq \frac{1}{\sqrt{2(n+k)}} + O(n^{-\beta'}), \tag{2.5}$$

for some integer k, where $\beta' > 1/2$.

Since (2.4) implies that $|\det DT(z)|^{-1} \leq 1 - 5|z|^2 + O(|z|^4)$, by (2.5) and Lemma 3.2 we get

$$|\det DT^{-n}(z)| \leq Dn^{-5/2}. \tag{2.6}$$

Also by (2.3),

$$DT^{-1}(x,y) = \begin{pmatrix} 1 - 3x^2 - y^2 + O(r^4) & -2xy + O(r^4) \\ -4xy + O(r^4) & 1 - 2x^2 - 6y^2 + O(r^4) \end{pmatrix}.$$

So $\|DT^{-1}(z)\| \leq 1 - |z|^2 + O(|z|^4)$, hence by Lemma 3.2,

$$\|DT^{-n}(z)\| \leq D'n^{-1/2} \tag{2.7}$$

for some $D' > 0$. Now by (2.6), (2.7) and (1.9), we know that $\delta = 5/2$ and $\tau = 1/2$. Since $m = 2$ and $\gamma = 2$, we have (1.8) if $\alpha = 1/2$. By Theorem B.I).ii), T satisfies Assumption 4(b).

Now we check that T satisfies Assumption 4(c). It is obvious that we can use Theorem B.II).i). However, we use this map to show how to apply Theorem B.II).ii).

Note that if we take two vectors $v_0 = (x,y)^*$ and $v_0' = (y, -x)^*$ at the tangent plane of $z = (x,y)$, where the asterisk denotes transpose, then by (2.3) we have

$$DT_z(v_0) = \begin{pmatrix} x + 3x^3 + 3xy^2 + O(|z|^5) \\ y + 6x^2y + 6y^3 + O(|z|^5) \end{pmatrix},$$

$$DT_z(v_0') = \begin{pmatrix} y + x^2y + y^3 + O(|z|^5) \\ -x - 2x^3 - 2xy^2 + O(|z|^5) \end{pmatrix}.$$

This means that $|DT_z(v_0')| < |DT_z(v_0)|$. We define $\mathcal{C}_z$ at each point z as the cone bounded by lines generated by vectors $3v_0 + 2v_0'$ and $3v_0 - 2v_0'$ and containing v_0, and define $\mathcal{C}_z'$ as the cone bounded by lines generated by vectors $3v_0' + 2v_0$ and $3v_0' - 2v_0$ and disjoint with $\mathcal{C}_z$. We can check that Part (a) and (b) in Theorem B.II).ii) are satisfied. Also we can check that for all unit vector $v' \in \mathcal{C}_z'$, $|DT_z(v')| \leq |Tz|^{2.5}/|z|^{2.5}$. So if we take R in such a way that the tangent lines of $\partial(T_1^{-n_0}R)$ are in the cones $\mathcal{C}'$ for some $n_0 \geq 0$, then we use the fact $DT^{-1}(\mathcal{C}') \subset \mathcal{C}'$ to get that Part (c) is satisfied for all $n \geq n_0$ with $1 - \theta = 2/5$.

In the next example the absolute continuous invariant measure μ has a finite and a σ-finite ergodic components simultaneously, and both contain the same indifferent fixed point p in their supports.

Example 4. *Suppose the map $T : M \to M$ satisfies Assumption 1 - 4(a), and in a neighborhood, say $B_1(p)$, of the indifferent fixed point p, T has the form as in (2.2).*

We also assume that there is a partition of $M = \{M_1, M_2\}$ such that for $i = 1, 2$, $TM_i = M_i$ and for any ball $B_\varepsilon(x)$ in M_i, there exists an integer N such that $T^N B_\varepsilon(x) = M_i$, and

$$\{z = (x, y) \in B_1(p) : y < x^2\} \subset M_1, \quad \{z = (x, y) \in B_1(p) : y > x^2\} \subset M_2.$$

This is possible since it is easy to check that $T\Gamma \cap B_1(p) = \Gamma$, where $\Gamma = \{(x, y) \in B_1(p) : y = x^2\}$.

By the above example, we know that T also satisfies Assumption 4(b) and 4(c). Therefore Theorem A can be applied. Since both M_1 and M_2 are invariant sets, T has absolutely continuous invariant measures μ_1 and μ_2 with respect to the Lebesgue measure restricted to M_1 and M_2 respectively. Now we show $\mu_1 M_1 < \infty$ and $\mu_2 M_2 = \infty$.

For this purpose we may assume that $R = B_1(p)$. By (2.5), we know that $T_1^{-n} R \subset B_{2/\sqrt{2n}}(p)$ for all large n. So

$$\nu(T_1^{-n} R \cap M_1) \le \nu\big\{(x, y) : x^2 + y^2 \le \frac{4}{2n},\ |y| \le |x|^2\big\} \le C\Big(\frac{4}{2n}\Big)^{3/2}$$

for some $C > 0$. It follows that $\sum_{n=1}^{\infty} \nu(T_1^{-n} R \cap M_1) < \infty$. Applying Theorem A to the system $T : M_1 \to M_1$, we get that $\mu_1 M_1 \le \infty$.

Also, by (2.5), we have that $T_1^{-n} R \supset B_{1/2\sqrt{4n}}(p)$ for all large n. Hence it is easy to see that $\nu(T_1^{-n} R) \ge \pi/16n$ and therefore $\sum_{n=1}^{\infty} \nu(T_1^{-n} R) = \infty$. Since $\nu\big(T_1^{-n} R \cap M_1\big) + \nu\big(T_1^{-n} R \cap M_2\big) = \nu\big(T_1^{-n} R\big)$, we get $\sum_{n=1}^{\infty} \nu(T_1^{-n} R \cap M_2) = \infty$. So we have $\mu_2 M_2 = \infty$.

3 Proof of Theorem B, Part I)

We first prove a few Lemmas.

For $\gamma > 0$, let $\beta = 1/\gamma$.

Lemma 3.1. *If*

$$t_{n-1} \ge t_n + C t_n^{1+\gamma} + O(t_n^{1+\gamma'}) \qquad \forall n > 0, \tag{3.1}$$

where $\gamma' > \gamma$, then for all large n,

$$t_n \le \frac{1}{(\gamma C(n+k))^\beta} + O\Big(\frac{1}{(n+k)^{\beta'}}\Big) \qquad \forall n > 0 \tag{3.2}$$

for some $\beta' > \beta$ and $k \in \mathbb{Z}$. The result remains true if we exchange "$\le$" and "$\ge$". Therefore, if (3.1) becomes an equality, then so does (3.2).

Proof: We claim that if

$$t_{n-1} \geq t_n + Ct_n^{1+\gamma} + C't_n^{1+\gamma'}, \tag{3.3}$$

for some large n and

$$t_n^\gamma \geq \frac{1}{\gamma Cn}\Big(1+\frac{1}{n^{\delta'}}\Big) \tag{3.4}$$

for some $\delta' > 0$, then

$$t_{n-1}^\gamma \geq \frac{1}{\gamma C(n-1)}\Big(1+\frac{1}{(n-1)^{\delta'}}\Big).$$

This gives the results since we can choose an integer k such that for some large $n_0 > 0$,

$$t_n^\gamma \leq \frac{1}{\gamma C(n_0+k)}\Big(1+\frac{1}{(n_0+k)^{\delta'}}\Big).$$

By relabelling the indices, the claim implies (3.2) for all $n \geq n_0$.

Now we prove the claim. Denote $\gamma_n = \gamma\big(1+n^{-\delta'}\big)^{-1}$. By (3.3) and (3.4),

$$t_{n-1}^\gamma \geq {t_n}^\gamma\big(1+C{t_n}^\gamma + C'{t_n}^{\gamma'}\big)^\gamma \geq \frac{1}{Cn\gamma_n}\Big(1+\frac{C}{Cn\gamma_n}+\frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma.$$

To prove the lemma we only need to show that

$$\frac{1}{n\gamma_n}\Big(1+\frac{1}{n\gamma_n}+\frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \geq \frac{1}{(n-1)\gamma_{n-1}},$$

or, equivalently,

$$\frac{n-1}{n}\Big(1+\frac{1}{n\gamma}+\frac{1}{n^{1+\delta'}\gamma}+\frac{C'}{(Cn\gamma_n)^{\gamma'/\gamma}}\Big)^\gamma \geq \frac{\gamma_n}{\gamma_{n-1}} = \frac{1+(n-1)^{-\delta'}}{1+n^{-\delta'}}.$$

Take $\delta' < \min\{1, \gamma'/\gamma - 1\}$. Then $(n\gamma_n)^{-(\gamma'/\gamma)}$ is of higher order. We can check that as $n \to \infty$, the left side of the inequality is like $1+n^{-(1+\delta')}$ and the right side is like $1+\delta' n^{-(1+\delta')}$. Since $\delta' < 1$, the right side is smaller as n large. $\square$

Lemma 3.2. *If for all $n > 0$, t_n satisfies (3.2), and $r(t_n) \leq 1 - C't_n^\gamma + O(t_n^{1+\gamma'})$, where $C' > 0$, then there exists $D > 0$ such that for all $k_0 \geq k$,*

$$\prod_{i=k_0-k}^{n+k_0-k} r(t_i) \leq D\Big(\frac{k}{n+k}\Big)^{C'/\gamma C}. \tag{3.5}$$

The result remains true if we replace "$\leq$" by "$\geq$" in all three inequalities.

Proof: Note that

$$r(t_n) \le 1 - \frac{C'}{\gamma C n} + O\Big(\frac{1}{n^{1+\gamma'}}\Big) = \Big(1 - \frac{1}{n}\Big)^{\frac{C'}{\gamma C}} \cdot \Big(1 + O\big(\frac{1}{n^{1+\gamma'}}\big)\Big),$$

where $\gamma' > 0$. Then we take the product. $\square$

Lemma 3.3. *Let $\theta \in (0,1)$ and $\bar{C}_1', \bar{C}_2, \bar{D}_1 > 0$, and let $\tilde{R} \subset \mathbb{R}^m$ be a bounded region containing the origin. Suppose the map $T : \tilde{R} \to \mathbb{R}^m$ is injective with $T^{-1}\tilde{R} \subset \tilde{R}$ and satisfies*

$$d(Tx, Ty) \ge (1 + \bar{C}_1'|x|^\gamma) d(x, y), \tag{3.6}$$

$$\log\Big|\frac{\det DT(x)}{\det DT(y)}\Big| \le \bar{C}_2 |x|^{\gamma-1} d(x, y) \tag{3.7}$$

for all $x, y \in \tilde{R}$ with $d(x, y) \le |x|/2$. Then there exists $J' > 0$ such that for all $x, y \in T\tilde{R}$ with

$$d(x_i, y_i)^{1-\theta} \le \bar{D}_1 |x_i|, \quad i = 1, \cdots, n, \tag{3.8}$$

where $x_i = T^{-i}x$ and $y_i = T^{-i}y$, we have

$$\log\Big|\frac{\det DT^n(x_n)}{\det DT^n(y_n)}\Big| \le J' d(x, y)^\theta. \tag{3.9}$$

Proof: We prove by induction that for all $i = 1, \cdots, n$,

$$\log\Big|\frac{\det DT^i(x_n)}{\det DT^i(y_n)}\Big| \le J' d(x_{n-i}, y_{n-i})^\theta. \tag{3.10}$$

For $i = 1$, by (3.7), (3.8) and (3.6), we have

$$\log\Big|\frac{\det DT(x_n)}{\det DT(y_n)}\Big| \le \bar{C}_2 \bar{D}_1 |x_n|^\gamma d(x_n, y_n)^\theta \le \bar{C}_2 \bar{D}_1 |x_{n-1}|^\gamma d(x_{n-1}, y_{n-1})^\theta.$$

So if $J' \ge \sup\{\bar{C}_2 \bar{D}_1 |x|^\gamma : \ x \in \tilde{R}\}$ then the right side of the inequality is less than $J' d(x_n, y_n)^\theta$ because $|x_n| \le |x|$.

Suppose (3.10) is true up to $i = k - 1$. Then similarly we have

$$\begin{aligned}
&\log\Big|\frac{\det DT^k(x_n)}{\det DT^k(y_n)}\Big| \le \log\Big|\frac{\det DT^{k-1}(x_n)}{\det DT^{k-1}(y_n)}\Big| + \log\Big|\frac{\det DT(x_{n-k+1})}{\det DT(y_{n-k+1})}\Big| \\
\le\ & J' d(x_{n-k+1}, y_{n-k+1})^\theta + \bar{C}_2 |x_{n-k+1}|^{\gamma-1} d(x_{n-k+1}, y_{n-k+1}) \\
=\ & J'\Big(1 + \frac{\bar{C}_2 \bar{D}_1}{J'} |x_{n-k+1}|^\gamma\Big) \cdot \frac{d(x_{n-k+1}, y_{n-k+1})^\theta}{d(x_{n-k}, y_{n-k})^\theta} \cdot d(x_{n-k}, y_{n-k})^\theta \\
\le\ & J'\Big(1 + \frac{\bar{C}_2 \bar{D}_1}{J'} |x_{n-k+1}|^\gamma\Big) \cdot \frac{1}{(1 + \bar{C}_1' |x_{n-k+1}|^\gamma)^\theta} d(x_{n-k}, y_{n-k})^\theta.
\end{aligned}$$

Clearly if J' is large enough, then the right side is bounded by $J'd(x_{n-k}, y_{n-k})^\theta$. We get (3.10) for $i = k$. $\square$

Proof of Theorem B, Part I):

i) We may assume that T is $C^{1+\hat{\alpha}}$ and $|\det DT| \geq \kappa^{-1} > 1$ on TR, because otherwise we can increase $N(\varepsilon)$ and J. We may also regard $\hat{\alpha} \leq 1$. So there exist $c_1 > 0$ such that

$$\frac{|\det DT_1^{-1}(y)|}{|\det DT_1^{-1}(x)|} \leq 1 + c_1 d(x, y)^{\hat{\alpha}}$$

for all $x, y \in TR$. Let $x_i = T_1^{-i}x$ and $y_i = T_1^{-i}y$. Clearly, $d(x_i, y_i) \leq d(x, y)$. So if $d(x, y) \leq \varepsilon$ and $0 < n \leq N$, then

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq \left(1 + c_1 d(x, y)^{\hat{\alpha}}\right)^n \leq \left(1 + c_1 \varepsilon)^{\hat{\alpha}}\right)^N \tag{3.11}$$

Also, there exists $C > 0$ such that for any $y \in B_\varepsilon(R_0)$, $|\det DT_1^{-n}(y)| \leq C\kappa^n$. Hence,

$$\sum_{n=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT_1^{-n}(y)| \leq \frac{C\kappa^N}{1 - \kappa}.$$

Let $b > 0$ be given.

Consider the function

$$\sigma(\varepsilon) = \frac{(1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon}}{1 + J\varepsilon^\alpha},$$

where $N_0 = 1 + \log(C^{-1}b(1 - \kappa))/\log \kappa$ and $c_2 = -(m + \alpha)/\log \kappa$. Since $\lim\limits_{\varepsilon \to 0}(1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon} = 1$, we have $\lim\limits_{\varepsilon \to 0} \sigma(\varepsilon) = 1$. Note that if

$$(N_0 - c_2 \log \varepsilon)\hat{\alpha}c_1\varepsilon^{\hat{\alpha}-1} \cdot (1 + J\varepsilon^\alpha) - \alpha J\varepsilon^{\alpha-1} \cdot (1 + c_1\varepsilon^{\hat{\alpha}}) < 0, \tag{3.12}$$

then $\sigma'(\varepsilon) < 0$. Since $\hat{\alpha} > \alpha$, the first term in (3.12) is of higher order. So we can choose $J > 0$ and $\varepsilon_4 > 0$ such that for all $\varepsilon \in (0, \varepsilon_4]$, (3.12) holds and therefore $\sigma(\varepsilon) \leq 1$.

Now for each $\varepsilon \in (0, \varepsilon_4]$, we take $N = N(\varepsilon)$ as the integer part of $N_0 - c_2 \log \varepsilon$. Clearly, for such N we have

$$\frac{C\kappa^N}{1 - \kappa} \leq b\varepsilon^{m+\alpha}.$$

So the second inequality in Assumption 4(b) is true. For the first inequality, note that

$$(1 + c_1\varepsilon^{\hat{\alpha}})^N \leq (1 + c_1\varepsilon^{\hat{\alpha}})^{N_0 - c_2 \log \varepsilon} \leq 1 + J\varepsilon^\alpha.$$

Then by (3.11) we get what we need.

ii) Denote $\beta = 1/\gamma$ and $\theta = \alpha$. Take $\rho > 0$ such that

$$\frac{\beta}{1-\theta} - \tau < \rho < \frac{\delta - 1}{m + \alpha}. \tag{3.13}$$

Let $b > 0$ be given.

Note that by Lemma 3.1, (1.5) implies that there exists $\bar{C}_0 > 0$ such that for any $x \in R_0$, $|x_n| \geq \dfrac{1}{(\bar{C}_0 n)^\beta}$. Take $N_b \geq L$ such that for all $n \geq N_b$,

$$b^{-\frac{1}{m+\alpha}} \Big(\sum_{k=n}^{\infty} \frac{C_\delta}{k^\delta} \Big)^{\frac{1}{m+\alpha}} < \frac{1}{n^\rho} < \frac{1}{(n-1)^\rho} < \frac{1}{2C_\tau \bar{C}_0^{\frac{\beta}{1-\theta}} n^{\frac{\beta}{1-\theta} - \tau}}, \tag{3.14}$$

where C_δ and C_τ are as in (1.9). The inequality is possible because of (3.13).

Note that (1.6) and (1.7) imply (3.6) and (3.7) respectively. By Lemma 3.3 we can take $J' > 0$ such that (3.9) holds for any $x \in R_0$, $n > 0$ whenever (3.8) holds with $\bar{D}_1 = 1$ for all x_i, y_i, $i = 1, \cdots, n$.

Take $\varepsilon_4' > 0$ such that for all x, y with $x \in R_0$, $d(x, y) \leq \varepsilon_4'$, $n = 1, \cdots, N_b$, we have $d(x_n, y_n)^{1-\theta} \leq |x_n|$. By the choice of J', (3.9) holds for all $1 \leq n \leq N_b$.

Then we take $\varepsilon_4 = \min\{\varepsilon_4', 1/N_b^\rho\}$, and $J > 0$ such that $e^{J' \varepsilon_4^\theta} \leq 1 + J\varepsilon_4^\theta$.

We show that J and ε_4 satisfies the requirement. Let $\varepsilon \in (0, \varepsilon_4]$. Take $N = N(\varepsilon) > N_b$ such that

$$\frac{1}{N^\rho} \leq \varepsilon < \frac{1}{(N-1)^\rho}.$$

By the first inequality of (1.9) and (3.14),

$$\sum_{k=N}^{\infty} \sup_{y \in B_\varepsilon(x)} |\det DT^{-k}(y)| \leq \sum_{k=N}^{\infty} \frac{C_\delta}{k^\delta} \leq b \cdot \frac{1}{N^{\rho(m+\alpha)}} \leq b\varepsilon^{m+\alpha}.$$

On the other hand, if $x \in R_0$ and $d(x, y) \leq \varepsilon$, then by the last inequality of (1.9) and (3.14), for any $N_b < n \leq N$,

$$d(x_n, y_n) \leq \frac{2C_\tau}{n^\tau} \varepsilon \leq \frac{2C_\tau}{n^\tau} \frac{1}{(N-1)^\rho} \leq \frac{1}{\bar{C}_0^{\frac{\beta}{1-\theta}} n^{\frac{\beta}{1-\theta}}} \leq |x_n|^{\frac{1}{1-\theta}}.$$

So we know that (3.9) holds for all $0 \leq n \leq N$. Then by the choice of J and the fact $\theta = \alpha$,

$$\Big| \frac{\det DT^n(x_n)}{\det DT^n(y_n)} \Big| \leq e^{J' d(x,y)^\theta} \leq e^{J' \varepsilon^\alpha} \leq 1 + J\varepsilon^\alpha.$$

This is what we need. $\square$

4 Proof of Theorem B, Part II)

This proof consist of two parts, i) and ii).

i) For $x \in \partial R$, denote

$$\mathcal{D}(x) = \{z \in R_0 : \ x \prec z \prec Tx\}.$$

Clearly the collection $\{\mathcal{D}(x) : \ x \in \partial R \cap \mathcal{C}_i\}$ form a cover of $\mathcal{C}_i \cap R_0$. So we can construct a partition ξ of R_0 such that every element of ξ belongs to some $\mathcal{D}(x)$.

Note that for any x,

$$\frac{|\det DT_1^{-n}(x)|}{|\det DT_1^{-n}(Tx)|} = \frac{|\det DT(x)|}{|\det DT(x_n)|} \leq |\det DT(x)|$$

is always bounded. So for any $y, z \in \mathcal{D}(x)$, we have

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(z)|} \leq \frac{|\det DT_1^{-n}(x)|}{|\det DT_1^{-n}(Tx)|} \leq |\det DT(x)|.$$

Hence (1.4) follows. Obviously we can arrange the partition ξ in such a way that (1.3) also holds. Therefore ξ is a desired partition for any n.

ii) First, we take $\theta > 0$ such that

$$DT_x(v') \leq (|Tx|/|x|)^{1/(1-\theta)}$$

for all $x \in \partial(T_1^{-n}R)$ and $v' \in E_x(\partial(T_1^{-n}R))$. This is possible because of the assumption stated in Part (d) of Theorem B.(II). So for any $n > 0$, if we take $x, y \in \partial R_0$ such that $d(x_n, y_n) \leq \bar{D}_1 |x_n|^{1/(1-\theta)}$, we have

$$d(x_i, y_i) \leq \bar{D}_1 |x_i|^{1/(1-\theta)} \qquad \forall i = 1, \cdots, n. \tag{4.1}$$

By Lemma 3.3, we get that there exists $I_1 > 0$ such that

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq I_1. \tag{4.2}$$

That is, (1.4) holds for all such x, y.

We construct $\xi = \xi_n$. Note that we only need do it for n sufficiently large. Since the family of cones $\mathcal{C}'_x$ are continuous uniformly in (t, ϕ), we can find $t_0 > 0$ such that for any $x, y \in TR$ with $d(x, y) \leq t_0$, the Hausdorff distance between $\mathcal{C}'_x$ and $\mathcal{C}'_y$ is less than $\theta_0/2$. Then we take $N > 0$ large enough such that for any $x \in R_0$ and $n > N$, $|x_n| \leq t_0$. Note that for any x, the position vector from p to x, denoted by u_x, is contained in $\mathcal{C}_x$. By Part (a) and (d) in the conditions of the theorem we know that at $x \in T_1^{-n}(\partial R_0)$, $\mathcal{C}'_x$ contains the tangent plane of the surface. Hence, if $v' \in E_x(T_1^{-n}(\partial R_0)$, then the angle between u_x and v', denoted by $\angle(u_x, v')$, is larger than θ_0, and therefore for any $v' \in E_y(T_1^{-n}(\partial R_0)$, we have $\angle(u_x, v') \geq \theta_0/2$, whenever $y \in T_1^{-n}(\partial R_0)$ with $d(x, y) \leq t_0$. So for any $x, y \in T_1^{-n}(\partial R_0)$ with $d(x, y) \leq t_0$, we have $d_S(x, y) \leq d(x, y)/\sin(\theta_0/2)$, where $d_S(\cdot, \cdot)$ is the distance restricted to the surfaces $\{T_1^{-n}(\partial R_0)\}$. This means that we can take a partition $\xi^{(n)}$ on $T_1^{-n}(\partial R_0)$ such that every element of $\xi^{(n)}$ is contained in a ball of radius $|x_n|^{1/(1-\theta)}$ and containing a ball of radius $|x_n|^{1/(1-\theta)}/10m\sin(\theta_0/2)$, with respect to the metric on $T_1^{-n}(\partial R_0)$, and these elements are close to $(m-1)$ dimensional disks. Denote $\xi' = T^n \xi^{(n)}$. Clearly, it is a partition of ∂R_0. Then we can take a partition ξ of R_0 whose elements has the form $\cup_{x \in A'} \mathcal{F}_x \cap R_0$, where A' is an element of ξ', and $\mathcal{F}_x$ is given in Lemma 4.1.

Now we prove that ξ satisfies (1.3) and (1.4). Condition (1.6) implies $\|DT(p)\| = 1$. We first consider the case that $DT(p) = \mathrm{id}$.

By (1.5), we know that $d(x, Tx) \leq C|x|^{1+\gamma}$ for some $C > 0$. So the "width" of the annulus $T_1^{-i}(B_{\varepsilon_5}(R_0))$ is bounded by $C'|T_1^{-i}x|^{1+\gamma}$ for some $C' > 0$. By Part (b) and (d) of the condition in the theorem, for $0 < \varepsilon \leq \varepsilon_5$, $x \in T_1^{-i}(\partial B_\varepsilon(R_0))$, the angle between the tangent space of $T_1^{-i}(\partial B_\varepsilon(R))$ and the position vector u_x is larger than θ_0. So the length of the curve $\mathcal{F}_{T_1^{-i}x} \cap T_1^{-i} B_{\varepsilon_5}(R_0)$ is bounded by $C|T_1^{-i}x|^{1+\gamma}$ for some $C \geq C'$. Hence, for any $x, y \in B_{\varepsilon_5}(R_0)$ with $y \in \mathcal{F}_x$, we can get

$$d(x_i, y_i) \leq C|x_i|^{1+\gamma} \tag{4.3}$$

and therefore by applying Lemma 3.3 get

$$\frac{|\det DT_1^{-n}(y)|}{|\det DT_1^{-n}(x)|} \leq I_2 \tag{4.4}$$

for some $I_2 > 0$. Also, the construction of ξ' implies (4.1) and therefore (4.2) for any $x, y \in A'$, where $A' \in \xi'$. So by the construction of ξ, we get (1.4) with $I = I_1 I_2^2$ for any $x, y \in A$.

On the other hand, for any $x, y \in B_{\varepsilon_5}(R_0)$ with $y \in \mathcal{F}_x$, we have (4.3). So we can apply Lemma 4.4 to get that inside A, distortion of $|DT|_{E(\mathcal{F})}|$ is bounded. It means that for each $x \in A$, the ratio of the length of $T_1^{-n}(B_\varepsilon(\partial R_0) \cap A)) \cap \mathcal{F}_{x_n}$ and the length of $T_1^{-n}(B_{\varepsilon_0}(\partial R_0) \cap A)) \cap \mathcal{F}_{x_n}$ is uniformly bounded by $\varepsilon/\varepsilon_0$ multiplied by a constant. Notice that the angle between the tangent vectors of $\mathcal{F}$ and the tangent space of $T_1^{-n}(\partial B_\varepsilon(R_0))$ are greater than θ_0. Also notice that by the construction of ξ', for any $A \in \xi$, the size of the set $T_1^{-n}A$ along the fiber direction is much smaller than the size of $T_1^{-n}A'$. Hence, the ratio between $\nu(T_1^{-n}B_\varepsilon(\partial R_0) \cap A))$ and $\nu(T_1^{-n}B_{\varepsilon_0}(\partial R_0) \cap A))$ is bounded by a constant times $\varepsilon/\varepsilon_0 \leq (\varepsilon/\varepsilon_0)^\alpha$ for some $\alpha \in (0, 1]$. Now we use (1.4) to get (1.3).‡

‡Let us make this argument more precise. We denote with A'_n and $A_n(\varepsilon)$ respectively the backward iterates $T_1^{-n}A'$ of some $A' \in \xi'$ and of the set $A \cap B_\varepsilon(\partial R_0)$ where $A = \cup_{x \in A'} \mathcal{F}_x \cap R_0$. Since the angles between the tangent spaces of the curves $\mathcal{F}_x$ and the tangent spaces of the ϵ-neighborhood of the boundary of R_0 are uniformly bounded away from zero, the length of

If $DT_p \neq \mathrm{id}$, then it is a rotation, say S. Hence near p we can write $Tx = Sx + T_r(x)$ where $|T_r(x)| \leq C|x|^{1+\gamma}$. If we write $T^{(i)} = \mathrm{id} + S^{-i} \circ T_r \circ S^{i-1}$, then $T^n = S^n \circ T^{(n)} \circ \cdots \circ T^{(1)}$. It implies that the "width" of the annulus $T_1^{-i} R_0$ is bounded by $C|T_1^{-i}x|^{1+\gamma}$. Then we apply the same arguments to get (1.4) and (1.3). $\square$

Lemma 4.1. *There is a foliation on $\{\mathcal{F}_x\}$ on $TR \setminus \{p\}$ consisting of curves from p to points on $\partial(TR)$ such that for any $x \in TR$, the tangent line of $\mathcal{F}_x$ lies in $\mathcal{C}_x$, and $T\mathcal{F}_x \cap TR = \mathcal{F}_{Tx}$.*

Proof: Denote $E_x = \cap_{n\geq 0} DT^n_{T_1^{-n}x}(\mathcal{C}_{T_1^{-n}x})$ for all $x \in TR\setminus\{p\}$. By Lemma 4.2, we know that sine of the angle between any two vectors in $DT^n_{T_1^{-n}x}(\mathcal{C}_{T_1^{-n}x})$ is less than $(1-d|x_n|^\gamma)\cdots(1-d|x_1|^\gamma)$. By (1.5) and Lemma 3.1, the product diverges as $n \to \infty$. So $\{E_x\}$ is a subbundle of the tangent bundle over $TR\setminus\{p\}$. Further, we have $DT_x(E_x) = E_{Tx}$ for all $x \in R$. By Lemma 4.3, we know that $\{E_x\}$ satisfies the Hölder condition near each x with Hölder constants depending on

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the curve $\mathcal{F}_x \cap B_\varepsilon(\partial R_0)$, when $x \in A'$, is of order ϵ. Its n-backward iterate in $A_n(\varepsilon)$ will be therefore bounded by a constant times ϵ times $d_{n,M}^{1+\gamma}$, where $d_{n,M}$ is the maximum over the ϵ-compact neighborhood of R_0 of $|T_1^{-i}x|$ (see above; equivalently we set $d_{n,m}$ the minimum of $|T_1^{-i}x|$ over the ϵ-compact neighborhood of R_0). Let us call this upper bound $l_{n,\epsilon}$. We construct then the $l_{n,\epsilon}$-neighborhood of A'_n, $B_{l_{n,\epsilon}}(A'_n)$. Clearly

$$\nu(A_n(\varepsilon)) \leq \nu(B_{l_{n,\epsilon}}(A'_n)) \leq \nu'(A'_{n,\varepsilon}) l_{n,\epsilon}$$

where $A'_{n,\varepsilon} = \{z \in T_1^{-n}A'; d(z, A'_n) \leq l_{n,\epsilon}\}$ and ν' denotes the riemannian volume on $T_1^{-n}\partial R_0$. Since $A'_{n,\varepsilon}$ is contained in a ball of radius $d_{n,M}^{\frac{1}{1-\theta}} + l_{n,\epsilon}$ and A'_n by construction contains a ball of radius $\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m}$, we have that $\nu'(A'_{n,\varepsilon}) \leq \mathrm{const}(d_{n,M}^{\frac{1}{1-\theta}} + l_{n,\epsilon})^{m-1}\gamma_{m-1}$ and $\nu'(A'_n) \geq (\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m})^{m-1}\gamma_{m-1}$. But $d_{n,M}, d_{n,m}$ are of order $n^{-\beta}$, with $\beta = 1/\gamma$ (see Lemma 3.1), and since $(1+\gamma)(1-\theta) > 1$, we see immediately that for large n:

$$\nu(A_n(\varepsilon)) \leq C'\nu'(A'_n) l_{n,\epsilon}$$

where C' is a suitable constant, depending on m. Let us now define the following objects: $A_n(\varepsilon_0)$: the backward iterate of $A \cap B_{\varepsilon_0}(\partial R_0)$, l'_{n,ϵ_0}: the minimum length of the backward images of the curves $\mathcal{F}_x \cap B_{\varepsilon_0}(\partial R_0)$, when $x \in A'$; $A'_{n,\varepsilon_0} = \{z \in A'_n; d(z, \partial A'_n) \geq l'_{n,\epsilon_0}\}$ and $B_{l''_{n,\varepsilon_0}}(A'_{n,\varepsilon_0})$ the l''_{n,ϵ_0}-neighborhood of A'_{n,ε_0}, being $l''_{n,\epsilon_0} = l'_{n,\epsilon_0}\sin\theta_0$. Moreover by what we already said above and which follows from Lemma 4.3, the bounded distortion property along the points of the backward images of the curves $A \cap B_{\varepsilon_0}(\partial R_0)$, will imply that l'_{n,ϵ_0} will be of the same order as l_{n,ϵ_0} (the maximum length of the backward images of the curves). Taking this into account we get:

$$\nu(A_n(\varepsilon_0)) \geq \nu(B_{l''_{n,\varepsilon_0}}(A'_{n,\varepsilon_0})) \geq ((\frac{d_{n,m}^{\frac{1}{1-\theta}} \sin\theta_0/2}{10m} - l''_{n,\epsilon_0})^{m-1}\gamma_{m-1} l''_{n,\epsilon_0}$$

By using as above the uniform bounds on $d_{n,M}, d_{n,m}$ when n is large, we see that $\nu(A_n(\varepsilon_0)) \geq C''\nu'(A'_n) l_{n,\epsilon_0}$, where C'' is a suitable constant depending on m. By dividing $\nu(A_n(\varepsilon))$ and $\nu(A_n(\varepsilon_0))$, we get the desired result.

x. Note that $\{E_x\}$ determines a vector field. We can integrate it to get a family of curves $\{\mathcal{F}_x\}$ from p to boundary points of TR that satisfies $T\mathcal{F}_x \cap TR = \mathcal{F}_{Tx}$. By our assumption, $\{\mathcal{F}_x\}$ is the "strong unstable manifold" at x.

It is easy to see that the curve passing through x is unique, and therefore $\{\mathcal{F}_x\}$ forms a foliation. In fact, if there are two such curves $\mathcal{F}_x$ and $\mathcal{F}'_x$ that pass through x, then we can take a curve Γ close to x joining $y \in \mathcal{F}_x$ and $y' \in \mathcal{F}'_x$ such that the tangent line of Γ is in $\mathcal{C}'$. Let us denote by A_n the area of the "triangle" bounded by the curves $T_1^{-n}\Gamma$, $T_1^{-n}\mathcal{F}_{x,y}$ and $T_1^{-n}\mathcal{F}'_{x,y'}$, and by L_n and L'_n the lengths of the curves $T_1^{-n}\mathcal{F}_{x,y}$ and $T_1^{-n}\mathcal{F}'_{x,y'}$ respectively, where $\mathcal{F}_{x,y}$ is the part of the curve in $\mathcal{F}_x$ between x and y, and $\mathcal{F}'_{x,y'}$ is understood in a similar way. By the assumption stated in Part (c), the ratio between A_n and $L_n \cdot L'_n$ tends to infinity, a contradiction. □

Lemma 4.2. *For any $v, v' \in \mathcal{C}_x$,*

$$\sin\angle(DT_x(v), DT_x(v')) \leq (1 - d|x|^\gamma) \sin\angle(v, v').$$

where the symbol $\angle(v, v')$ denotes the angle between the vectors v and v'.

Proof: Note that

$$|\det DT_x|_{E(v,v')}| = \frac{|DT_x(v)| \cdot |DT_x(v')| \cdot \sin\angle(DT_x(v), DT_x(v'))}{|v| \cdot |v'| \cdot \sin\angle(v, v)}$$

and

$$||DT_x|_{E(v)}|| = \frac{|DT_x(v)|}{|v|}, \qquad ||DT_x|_{E(v')}|| = \frac{|DT_x(v)|}{|v'|}.$$

Then the results follows from (1.10). □

Lemma 4.3. *There exist constants $H > 0$, $a > 0$, and $\tau_1 \in (0, 1)$, such that for all $x \in TR\backslash\{p\}$,*

$$d(E_x, E_y) \leq \frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} \qquad \forall y \in B(x, a|x|), \tag{4.5}$$

where $d(E_x, E_y)$ is defined by $d(E_x, E_y) = \sin\angle(v_x, v_y)$, v_x and v_y are the tangent vectors of $\mathcal{F}_x$ and $\mathcal{F}_y$ at x and y respectively chosen in the way that $0 \leq \angle(v_x, v_y) < \pi/2$.

Proof: We note that we only need prove (4.5) for all x in a small neighborhood $\tilde{R} \subset R$ of p, because $DT_x(E_x) = E_{Tx}$, and then the results can be extended to TR.

Take $\tilde{d} \in (0, d)$. Then for each x we can extend $\mathcal{C}_x$ to $\tilde{\mathcal{C}}_x$ such that (1.10) hold with $\tilde{d}$ for all $v \in \mathcal{C}_x$ and $v' \in \tilde{\mathcal{C}}_x$. By (1.5) and the fact that T is $C^{1+\gamma}$, we can write $DT(x) = T_0(x) + T_\gamma(x) + T_h(x)$, where $T_0 = DT_p$, T_γ satisfies $T_\gamma(tx) = t^\gamma T_\gamma(x)$ $\forall t > 0$ and $|T_h(x)| = o(|x|^\gamma)$. So it is easy to see that we can find $\varepsilon_a > 0$ such that $\tilde{\mathcal{C}}_x \cap \mathbb{S}^{m-1}$ contains an ε_a-neighborhood of $\mathcal{C}_x \cap \mathbb{S}^{m-1}$ in $\mathbb{S}^{m-1}$ for all x with $|x|$ small. Moreover, since $\mathcal{C}_x$ is uniformly continuous in (t, ϕ), we can take $a > 0$ and $\tilde{R}$ small such that for all $x \in \tilde{R}$, with $d(x, y) \le a|x|^\gamma$, $\mathcal{C}_y \subset \tilde{\mathcal{C}}_x$. So if $v \in \mathcal{C}_x$ and $v' \in \mathcal{C}_y$, we have

$$\frac{|\det DT_x|_{E(v,v')}|}{\|DT_x|_{E(v)}\| \cdot \|DT_x|_{E(v')}\|} \le 1 - \tilde{d}|x|^\gamma.$$

Hence, by the same arguments used in Lemma 4.2 we have

$$\sin\angle(DT_x(v), DT_x(v')) \le (1 - \tilde{d}|x|^\gamma)\sin\angle(v, v'). \tag{4.6}$$

Take $\tau_1 \in (0, 1)$ such that

$$\Big(1 - \frac{\tilde{d}}{2}|x|^\gamma\Big)\Big(\frac{|Tx|}{|x|} \cdot \frac{d(x, y)}{d(Tx, Ty)}\Big)^{\tau_1} \le 1 \tag{4.7}$$

for all $x \in \tilde{R}$ close to p with $d(x, y) \le a|x|$.

Take $0 < a_1 \le a$ such that if $d(x, y) \le a_1|x|$, then

$$\|DT(x) - DT(y)\| \le \bar{C}_2|x|^{\gamma-1}d(x, y)^{\tau_1}. \tag{4.8}$$

for some $\bar{C}_2 > 0$. This is possible because of (1.7).

Take $H > 0$ such that $H\tilde{d} > 2\bar{C}_2$.

Let $\mathcal{L} = \{L_x : x \in \tilde{R}\backslash\{p\}\ \}$ be the set of all line bundles in the tangent bundle over $\tilde{R}$. Clearly DT induces a map $\mathcal{D} : \mathcal{L} \to \mathcal{L}$ given by $(\mathcal{D}\mathcal{L})_x = DT_x(L_{T_1^{-1}x})$, and $\mathcal{E} = \{E_x\}$ is the unique fixed point of $\mathcal{D}$ contained in $\mathcal{C}$. Denote

$$\mathcal{H} = \Big\{\{L_x\} \in \mathcal{L} \cap \mathcal{C} : d(L_x, L_y) \le \frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} \qquad \forall y \in B(x, a_1|x|)\ \Big\}. \tag{4.9}$$

We show that $\mathcal{D}(\mathcal{H}) \subset \mathcal{H}$. This implies the result since $\{E_x\} = \cap_{n\ge0}\mathcal{D}^n\mathcal{C}$.

Take $\{L_x\} \in \mathcal{H}$. Let $x, y \in \tilde{R}$ with $d(x, y) \le a_1|x|$. Take unit vectors $e_x \in L_x$, $e_y \in L_y$. So $\sin\angle(e_x, e_y) \le H|x|^{-\tau_1}d(x, y)^{\tau_1}$. By (4.6) and (4.8),

$$\begin{aligned}
& \sin\angle(DT_x(e_x), DT_y(e_y)) \\
\le\ & \sin\angle(DT_x(e_x), DT_x(e_y)) + \sin\angle(DT_x(e_y), DT_y(e_y)) \\
\le\ & (1 - \tilde{d}|x|^\gamma)\sin\angle(e_x, e_y) + |DT_x(e_y) - DT_y(e_y)| \\
\le\ & (1 - \tilde{d}|x|^\gamma)\frac{Hd(x, y)^{\tau_1}}{|x|^{\tau_1}} + \bar{C}_2|x|^{\gamma-1}d(x, y)^{\tau_1} \\
=\ & \big[(1 - \tilde{d}|x|^\gamma)H + \bar{C}_2|x|^\gamma\big]\frac{d(Tx, Ty)^{\tau_1}}{|Tx|^{\tau_1}} \cdot \frac{d(x, y)^{\tau_1}}{d(Tx, Ty)^{\tau_1}}\frac{|Tx|^{\tau_1}}{|x|^{\tau_1}}.
\end{aligned}$$

By the choice of H, the quantity in the blanket is less than $1 - \tilde{d}|x|^\gamma/2$. Then by (4.7) the right side of the inequality is less than $H|Tx|^{-\tau_1}d(Tx, Ty)^{\tau_1}$. We get the desired results. □

Lemma 4.4. *There exists $J^* > 0$ such that for any x, y with $d(x_i, y_i) \le |x_i|^{\bar\gamma}$ for some $\bar\gamma > 1$, $i = 1, \cdots, n$,*

$$\frac{|DT_1^{-n}(y)|_{E_y(\mathcal{F})}|}{|DT_1^{-n}(x)|_{E_x(\mathcal{F})}|} \le J^*. \tag{4.10}$$

Proof: Take an integer $\bar r \ge 2C_0'/C_0$, where C_0 and C_0' are as in (1.6). We assume that $x_0 \le 1/(\gamma C_0' k_0)^\beta$ for some $k_0 \ge 1$. Then we take $k_i = (\bar r^i - 1)k_0$ for $i = 1, \cdots, \ell - 1$, where $\ell - 1$ is the largest number j such that $k_j < n$. Let $k_\ell = n$. By Lemma 3.1, we know that

$$|x_j|^\gamma \le 2/(\gamma C_0'(k_0 + j)). \tag{4.11}$$

Hence, (1.6) implies

$$\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \le \prod_{j=k_{i-1}}^{k_i - 1} \|DT_{x_j}\| \le \prod_{j=k_{i-1}}^{k_i - 1} \Big(1 + \frac{2C_1}{\gamma C_0'(k_0 + j)}\Big) \le \prod_{j=k_{i-1}}^{k_i - 1} \Big(1 + \frac{1}{k_0 + j}\Big)^C$$

for some C larger than $2C/\gamma C_0'$ if k_i is large enough. So the choice of $\bar r$ gives

$$\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \le \Big(\frac{k_0 + k_i}{k_0 + k_{i-1}}\Big)^C \le \bar r^C \tag{4.12}$$

for all $i \ge 0$.

Let e_x be the unit tangent vector of $\mathcal{F}$ at x. We have

$$\begin{aligned}
\frac{|DT_1^{-n}(y)|_{E_y(\mathcal{F})}|}{|DT_1^{-n}(x)|_{E_x(\mathcal{F})}|} &= \frac{|DT_{x_n}^n(e_{x_n})|}{|DT_{y_n}^n(e_{y_n})|} = \frac{|DT_{x_n}^n(e_{x_n})|}{|DT_{x_n}^n(e_{y_n})|} \cdot \frac{|DT_{x_n}^n(e_{y_n})|}{|DT_{y_n}^n(e_{y_n})|} \\
&= \prod_{i=1}^{\ell} \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \cdot \prod_{j=1}^{n} \frac{|DT_{x_j}(e_{y_j})|}{|DT_{y_j}(e_{y_j})|}.
\end{aligned}$$

By the results of Lemma 4.3 and (4.12), each factor in the first product is bounded by

$$\begin{aligned}
& 1 + \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}})| - |DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \\
\le\ & 1 + \frac{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{x_{k_i}} - e_{y_{k_i}})|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \le 1 + \frac{\|DT_{x_{k_i}}^{k_i - k_{i-1}}\| \cdot |e_{x_{k_i}} - e_{y_{k_i}}|}{|DT_{x_{k_i}}^{k_i - k_{i-1}}(e_{y_{k_i}})|} \\
\le\ & 1 + \frac{\bar r^C \cdot BHd(x_{k_i}, y_{k_i})^{\tau_1}}{|x_{k_i}|} \le 1 + \bar r^C BH |x_{k_i}|^{\tau_1(\bar\gamma - 1)},
\end{aligned}$$

where we use the fact that $|e_{x_{k_i}} - e_{y_{k_i}}| \le B \sin\angle(e_{x_{k_i}}, e_{y_{k_i}})$ for some $B > 0$. Also note that by (4.11) and the choice of k_i, $\{|x_{k_i}|\}$ decreases exponentially fast as $i \to \infty$. Since $\bar\gamma > 1$, the first product in above equality is convergent.

For the second product, by (1.7) each factor is bounded by

$$1+\frac{|DT_{x_j}(e_{y_j})|-|DT_{y_j}(e_{y_j})|}{|DT_{y_j}(e_{y_j})|}\le 1+\frac{C|x_j|^{\gamma-1}d(x,y)}{|DT_{y_j}(e_{y_j})|}\le 1+\frac{C|x_j|^{\bar\gamma+\gamma-1}}{|DT_{y_j}(e_{y_j})|}.$$

By (4.11) and the fact $\bar\gamma>1$, we know that $\sum_j |x_j|^{\bar\gamma+\gamma-1}$ converges. So the product is also bounded. We get the result. □

5 Proof of Theorem A

In this section we first introduce a subspace V_α of $L^1\equiv L^1(\mathbb{R}^m,\nu)$ with compact unit ball that contains the density function of the invariant measures of the induced map of T with respect to the relatively compact subspace $M\backslash R$. Here we only give a brief description and list some properties we use. We refer to [17] and [13] for more details.

Let f be an $L^1(\mathbb{R}^m,\nu)$ function. If Ω is a Borel subset of $\mathbb{R}^m$, we define the oscillation of f over Ω by the difference of essential supremum and essential infimum of f over Ω:

$$\operatorname{osc}(f,\Omega)=\operatorname*{Esup}_{\Omega} f-\operatorname*{Einf}_{\Omega} f.$$

If $B_\epsilon(x)$ denotes the ball of radius ϵ about the point x, then we get a measurable function $x\to\operatorname{osc}(f,\ B_\epsilon(x))$. The function have the following properties.

Proposition 5.1. *Let $f,f_i,g\in L^\infty(\mathbb{R}^m,\nu)$ with $g\ge 0$, $\varepsilon>0$, and S be a Borel subset of $\mathbb{R}^m$. Then*

(i) $\operatorname{osc}\big(\sum_i f_i,\ B_\varepsilon(\cdot)\big)\le\sum_i\operatorname{osc}\big(f_i,\ B_\varepsilon(\cdot)\big)$,

(ii) $\operatorname{osc}\big(f\chi_S,\ B_\varepsilon(\cdot)\big)\le\operatorname{osc}\big(f,\ S\cap B_\varepsilon(\cdot)\big)\chi_S(\cdot)+2\big[\operatorname*{Esup}_{S\cap B_\varepsilon(\cdot)} f\big]\chi_{B_\varepsilon(S)\cap B_\varepsilon(S^c)}$,

(iii) $\operatorname{osc}\big(fg,\ S\big)\le\operatorname{osc}\big(f,\ S\big)\operatorname*{Esup}_S g+\operatorname{osc}\big(g,\ S\big)\operatorname*{Einf}_S f$.

Proof: See [17] Proposition 3.2. □

Take $0<\alpha<1$ and $\varepsilon_0>0$. We define the α-seminorm of f as:

$$|f|_\alpha=\sup_{0<\epsilon\le\epsilon_0}\epsilon^{-\alpha}\int_{\mathbb{R}^m}\operatorname{osc}(f,B_\epsilon(x))d\nu(x). \tag{5.1}$$

We will consider the space of the functions f with bounded α-seminorm, namely:

$$V_\alpha=\big\{f\in L^1:|f|_\alpha<\infty\big\} \tag{5.2}$$

and equip V_α with the norm:

$$\| \cdot \|_\alpha = \| \cdot \|_1 + | \cdot |_\alpha, \tag{5.3}$$

where $\| \cdot \|_1$ denotes the L^1 norm. This space will not depend on the choice of ϵ_0. With the $\| \cdot \|_\alpha$ norm, V_α is a Banach space; moreover according to Theorem 1.13 in [13], the unit ball in V_α is compact in L^1.

Proposition 5.2. *Let $f \in V_\alpha$; then:*

(i) $\|f\|_\infty \le \dfrac{1}{\gamma_m \varepsilon_0^m} \|f\|_\alpha$ *provided* $\varepsilon_0 \le 1$.

(ii) *There exists a ball $B_\varepsilon(x)$ such that* $\operatorname*{Einf}_{B_\varepsilon(x)} f > 0$.

Proof: See [17] Proposition 3.4 and Lemma 3.1. □

To prove Theorem A we need one more ingredient, the so-called Lasota-Yorke's inequality, which will be proved in Section 6. This inequality provides an upper bound on the action of the Perron-Frobenius operator on the elements on V_α. Such an operator will be defined on the subspace $M \backslash R$ with a potential given by the inverse of the determinant of the induced map. We will denote it as $\hat{P} f$. We will prove that:

$$|\hat{P} f|_\alpha \le \eta |f|_\alpha + D \|f\|_1$$

where $\eta < 1$ and $D < \infty$. This, plus the compactness in L^1 of the unit ball of V_α, will allow us to invoke the ergodic theorem of Ionescu-Tulcea and Marinescu [12] (see also [13], Theorem 3.3), to conclude that there exists a unique (greatest)[§] invariant probability measures μ which is absolutely continuous with respect to ν on $M \backslash R$ and which decomposes into a finite number of cyclic disjoint measurable sets upon which a certain power of the map is mixing.

Proof of Theorem A:

Recall that R is given in Assumption 3. We construct a induced system $(\hat{M}, \hat{T})$. Denote $\hat{M} = M \backslash R$. Let $\hat{T} : \hat{M} \to \hat{M}$ be the first return map of T, so that $\hat{T}(x) = T(x)$ if $x \notin T^{-1}R$, otherwise $\hat{T}(x) = T^{i+1}(x) = T_1^i T_j(x)$ if $x \in T_j^{-1} R$, where i is the smallest positive integer such that $T_1^i T_j(x) \notin R$. We denote $g(x) = |\det DT(x)|^{-1}$, and similarly $\hat{g}(x) = g(x)$ if $x \notin T^{-1}R$ and $\hat{g}(x) = |\det DT^{i+1}(x)|^{-1}$ if otherwise. Let $\hat{\nu}$ be the conditional measure of the Lebesgue measure ν. We may still think that $\hat{\nu}$ is a Lebesgue measure with $\hat{\nu}(\hat{M}) = 1$.

Let P be the Perron-Frobenius operator of T with the potential function $\log g(x)$, i.e.

$$Pf(x) = \sum_{Ty=x} f(y) g(y).$$

[§] "Unique greatest" means that any other measure absolutely continuous with respect to ν is absolutely continuous with respect to μ.

Then let $\hat{P}$ be the Perron-Frobenius operator of $\hat{T}$ with the potential function $\log \hat{g}(x)$.

By Proposition 6.2 in the next section we have the Lasota-Yorke's inequality for the induced system $(\hat{M}, \hat{T})$. So $\hat{T}$ has an absolutely continuous invariant probability measure $\hat{\mu}$ on $\hat{M}$ with density function $\hat{h}$ that has finitely many ergodic components.

We extend $\hat{\mu}$ to M to get an invariant measure of T. Recall $R_0 = TR\backslash R$, and let $R_n = T_1^{-n} R_0$ for $n > 0$. By Remark 1.1, $\operatorname{diam} R_n \to 0$. So we have $R = \sum_{n=1}^{\infty} R_n \cup \{p\}$. We extend $\hat{h}$ to R to get a density function h on M. That is, if h is defined on $M \backslash T_1^{-n} R$, then for $x \in T_1^{-n} R \backslash T_1^{-n-1} R$, we let

$$h(x) = g(x)^{-1} \cdot \Big(h(Tx) - \sum_{j \neq 1} h(T_j^{-1} Tx) g(T_j^{-1} Tx) \Big).$$

It is easy to see that $h \geq 0$ and $Ph = h$ on M. Let μ be the measure on M with density h. Clearly, μ is invariant under T and has the same number of ergodic components as $\hat{\mu}$ does.

Next, we show that μM is finite if $\sum\limits_{i=1}^{\infty} \nu(T_1^{-i} R) < \infty$. Since μ is invariant, we have

$$\mu R_i = \mu R_{i+1} + \sum_{j=2}^{K'} \mu(T_j^{-1} R_i),$$

where we assume that in addition to $T_1^{-1} R \subset U_1$, R has $K' - 1$ preimages in $U_2, \cdots, U_{K'}$, where $K' \leq K$. Take summation from $i = n$ to infinity, we get

$$\mu R_n = \sum_{j=2}^{K'} \mu\big(T_j^{-1} \bigcup_{i=n}^{\infty} R_i\big) = \sum_{j=2}^{K'} \mu\big(T_j^{-1} T_1^{-n} R\big).$$

Note that $\|\hat{h}\|_\infty \leq \infty$ since $\hat{h} \in V_\alpha$, and then note that the Jacobian of T_j^{-1} is less than or equal to 1. We have

$$\mu\big(T_j^{-1} T_1^{-n} R\big) \leq \|\hat{h}\|_\infty \nu\big(T_j^{-1} T_1^{-n} R\big) \leq \|\hat{h}\|_\infty \nu\big(T_1^{-n} R\big).$$

Hence

$$\mu R = \sum_{n=1}^{\infty} \mu R_n \leq \|\hat{h}\|_\infty (K' - 1) \sum_{n=1}^{\infty} \nu\big(T_1^{-n} R\big) < \infty. \tag{5.4}$$

Now we prove the last part of the theorem. By Proposition 5.2(ii), there is a ball $B_\varepsilon(z) \subset M \setminus R$ such that $\operatorname*{Einf}\limits_{B_\varepsilon(x)} \hat{h} \geq h_* > 0$ for some constant h_*. By our assumption, there exists $\tilde{N} > 0$ such that $T^{\tilde{N}} B_\varepsilon(z) \supset M$. So for any $x \in M$, there is $y_0 \in B_\varepsilon(z)$ such that $T^{\tilde{N}} y_0 = x$. Since $|\det DT|$ is bounded above, we have $g_* := \inf\{g(y) : \ y \in M\} > 0$. Hence, for every x,

$$h(x) = (P^{\tilde{N}}h)(x) = \sum_{T^{\tilde{N}}y=x} h(y) \prod_{i=0}^{\tilde{N}-1} g(T^i y) \geq h(y_0) \prod_{i=0}^{\tilde{N}-1} g(T^i y_0) \geq h_* g_*^{\tilde{N}}.$$

In this case, we can use a similar method as for (5.4) to get

$$\mu R = \sum_{n=1}^{\infty} \mu R_n \geq (h_* g_*^{\tilde{N}}) g_* \sum_{j=2}^{K'} \nu\big(T_1^{-n} R\big) = \infty.$$

This ends the proof. $\square$

6 A Lasota-Yorke type inequality

Let R be as in Assumption 3. Denote $\hat{T}_{ij} = T_1^i T_j$ and $U_{ij} = \hat{T}_{ij}^{-1}(R_0) = T_j^{-1} R_i$ for $i > 1$ and $U_{0j} = U_j \backslash T_j^{-1} R$. So if $TU_l \not\ni p$, then U_{il} is undefined for any $i > 0$ and $U_{0l} = U_l$. Clearly, $U_{ij} \subset U_j$ for all $i > 0$ and $\{U_{ij}, i \geq 0\}$ are pairwise disjoint.

Lemma 6.1. *There exists $0 < \varepsilon_6 \leq \varepsilon_5$ such that for any $\varepsilon_0 \leq \varepsilon_6$, $\varepsilon \leq \varepsilon_0$, $x \in M$,*

$$2 \sum_{j=1}^{K} \sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))} \leq \frac{\lambda \varepsilon^\alpha}{\varepsilon_0^\alpha}, \tag{6.1}$$

where λ is given by Assumption 3(b).

Proof: Note that the sets $\cup_{i=1}^{\infty} \partial U_{ij}$, $j = 1, \cdots, K$, are pairwise separated. So by Assumption 3(b) and the definition λ in (1.1) we only need prove that there exists $\varepsilon_6 > 0$ such that for any given j, for any x in the ε_6-neighbourhood of $T_j^{-1} R_0$, if $0 < \varepsilon \leq \varepsilon_0 \leq \varepsilon_6$, then

$$2 \sum_{i=0}^{\infty} \frac{\nu(\hat{T}_{ij}^{-1} B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_6}(x))}{\nu(B_{(1-s)\varepsilon_6}(x))} \leq \frac{\lambda \varepsilon^\alpha}{\varepsilon_0^\alpha}. \tag{6.2}$$

Take

$$\varepsilon_6 \leq \min\{\varepsilon_5, \varepsilon_3\} \cdot \Big(\frac{\lambda(1-s)^m}{2C_\xi I^2}\Big)^{1/\alpha},$$

where ε_3 is given by Assumption 3(c).

Recall that N_s is also given by Assumption 3(c). Reduce ε_6 if necessary such that for any x, the ball $B_{(1-s)\varepsilon_6}(x)$ intersects at most one connected component of the set $\{\hat{T}_{ij}^{-1}B_{\varepsilon_6}(\partial R_0), 0 < i \le N, 1 < j \le K\}$. We also require ε_6 small enough such that for any $1 < j \le K$, $1 \le i \le N_s$, the part $\hat{T}_{ij}^{-1}\partial R_0 \cap B_{\varepsilon_6}(x)$ are close to an $(m-1)$ dimensional plane.

Take ε and ε_0 such that $0 < \varepsilon \le \varepsilon_0 \le \varepsilon_6$.

We first consider the case $1 \le i \le N_s$. Note that $\hat{T}_{ij}^{-1}B_\varepsilon(\partial R_0) \cap B_{(1-s)\varepsilon_0}(x) \subset B_{s\varepsilon}(T_{ij}^{-1}\partial R_0) \cap B_{(1-s)\varepsilon_0}(x)$. The volume of the latter is close to $\gamma_{m-1}((1-s)\varepsilon_0)^{m-1} \cdot 2s\varepsilon = 2s\gamma_{m-1}\varepsilon(1-s)^{m-1}\varepsilon_0^{m-1}$. So $\dfrac{\nu(\hat{T}_{ij}^{-1}B_\varepsilon(\partial R) \cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))}$ is close to $\dfrac{2s\gamma_{m-1}\varepsilon(1-s)^{m-1}\varepsilon_0^{m-1}}{\gamma_m(1-s)^m\varepsilon_0^m} = \dfrac{2s\gamma_{m-1}\varepsilon}{(1-s)\gamma_m\varepsilon_0}$. Hence, by Assumption 3(b), we know that it is less than $\lambda\varepsilon^\alpha/\varepsilon_0^\alpha$.

Now we consider the case that $i \ge N_s$.

Let $\tilde{\varepsilon} = \varepsilon_0\Big(\dfrac{2C_\xi I^2}{\lambda(1-s)^m}\Big)^{1/\alpha}$. we have $\tilde{\varepsilon} \le \varepsilon_5$.

For each i, we take a partition $\xi_i = \{\tilde{A}_{i1}, \tilde{A}_{i2}, \cdots, \}$ satisfying Assumption 4(c) with $n = i$ and $\tilde{\varepsilon} \le \varepsilon_5$. Denote $A_{ik} = \tilde{A}_{ik} \cap B_{\tilde{\varepsilon}}(\partial R_0)$, $A'_{ik} = \tilde{A}_{ik} \cap B_\varepsilon(\partial R_0)$, $A_{ijk} = \hat{T}_{ij}^{-1}A_{ik}$ and $A'_{ijk} = \hat{T}_{ij}^{-1}A'_{ik}$. Then we let

$$\mathcal{A} = \{A_{ijk} : \ A'_{ijk} \cap B_{(1-s)\varepsilon_0}(x) \ne \emptyset\}, \qquad \mathcal{A}' = \{A'_{ijk} : \ A_{ijk} \in \mathcal{A}\}.$$

By abusing notations, we may also think that $\mathcal{A}$ and $\mathcal{A}'$ are the unions of the sets they contain.

By the fact

$$\nu A_{ijk} = \int_{A_{ik}} |\det D\hat{T}_{ij}^{-1}(x)| d\nu(x)$$

and Assumption 4(c), we know that

$$\frac{\nu A'_{ijk}}{\nu A_{ijk}} \le \frac{C_\xi \varepsilon^\alpha}{\tilde{\varepsilon}^\alpha} \cdot I^2 = \frac{C_\xi I^2 \varepsilon^\alpha \lambda(1-s)^m}{2C_\xi I^2 \varepsilon_0^\alpha} = \frac{\varepsilon^\alpha\lambda(1-s)^m}{2\varepsilon_0^\alpha}. \tag{6.3}$$

Denote $s^* = \sup\big\{s(T_1^{-N_s}(z), T_1^{N_s}) : \ z \in B_{\tilde{\varepsilon}}(R_0)\big\}$. Note that by Assumption 4(c), $\operatorname{diam} A_{ik} \le 5m\tilde{\varepsilon} \le 5m\varepsilon_0\Big(\dfrac{2C_\xi I^2}{\lambda(1-s)^m}\Big)^{1/\alpha}$. Since $i \ge N_s$, by Assumption 3(c), we have

$$\operatorname{diam} A_{ijk} \le 5m\varepsilon_0\Big(\frac{2C_\xi I^2}{\lambda(1-s)^m}\Big)^{1/\alpha} \cdot s^* = s\varepsilon_0. \tag{6.4}$$

So if $A_{ijk} \in \mathcal{A}$, then $A_{ijk} \cap B_{(1-s)\varepsilon_0}(x) \ne \emptyset$, and therefore $A_{ijk} \subset B_{\varepsilon_0}(x)$. That is,

$$\mathcal{A} \subset B_{\varepsilon_0}(x). \tag{6.5}$$

Note that

$$\bigcup_{i=0}^{\infty} \hat{T}_{ij}^{-1}B_\varepsilon(\partial R) \cap B_{(1-s)\varepsilon_0}(x) \subset \mathcal{A}'. \tag{6.6}$$

By (6.3)-(6.6), we get

$$2\sum_{i=0}^{\infty}\frac{\nu(\hat{T}_{ij}^{-1}B_{\varepsilon}(\partial R)\cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))}\le 2\cdot\frac{\nu\mathcal{A}'}{\nu\mathcal{A}}\cdot\frac{\nu\mathcal{A}}{\mu B_{\varepsilon_0}(x)}\cdot\frac{\mu B_{\varepsilon_0}(x)}{\nu(B_{(1-s)\varepsilon_0}(x)}$$
$$\le\quad 2\cdot\frac{\varepsilon^{\alpha}\lambda(1-s)^m}{2\varepsilon_0^{\alpha}}\cdot 1\cdot\frac{\gamma_m\varepsilon_0^m}{\gamma_m(1-s)^m\varepsilon_0^m}=\lambda\frac{\varepsilon^{\alpha}}{\varepsilon_0^{\alpha}}.$$

This is (6.2), the formula we need show. □

Proposition 6.2. *Assume that $T: M\to M$ satisfies Assumption 1-4, and $\hat{T}:\hat{M}\to\hat{M}$ is the reduced system with respect to $\hat{M}=M\backslash R$. Then there exist $\eta<1$ and $D<\infty$ such that for any $f\in V_{\alpha}=V_{\alpha}(\varepsilon_0)$, we have $Pf\in V_{\alpha}$ and*

$$|\hat{P}f|_{\alpha}\le\eta|f|_{\alpha}+D\|f\|_1$$

for all ε_0 sufficiently small.

Proof: Take $\zeta>0$ such that for any $\varepsilon\le\varepsilon_4$,

$$(1+Js^{\alpha}\varepsilon^{a})(1+cs^{a}\varepsilon^{\alpha})\le 1+\zeta\varepsilon^{\alpha},\tag{6.7}$$

where c and J, ε_4 and are given in Assumption 4(a) and (b) respectively.

Recall that by Assumption 3(b), $s^{\alpha}+\lambda\le\eta_0<1$. Take $b>0$ such that $(s^{\alpha}+\lambda)+3K'b\gamma_m^{-1}<1$, where K' is the number of preimages of p for the map T. Recall also that $\varepsilon_1,\varepsilon_2$, ε_4 and ε_6 are given in Assumption 1(b), 3(b) and 4(b) and Lemma 6.1 respectively. Take $\varepsilon_0\le\min\{\varepsilon_1,\varepsilon_2,\varepsilon_4,\varepsilon_6\}$ such that

$$\eta:=(1+\zeta\varepsilon_0^{\alpha})(s^{\alpha}+\lambda)+3K'b\gamma_m^{-1}<1.\tag{6.8}$$

Denote

$$G_R(x,\varepsilon,\varepsilon_0)=2\sum_{j=1}^{K}\sum_{i=0}^{N(\varepsilon)}\frac{\nu(\hat{T}_{ij}^{-1}B_{\varepsilon}(\partial R_0)\cap B_{(1-s)\varepsilon_0}(x))}{\nu(B_{(1-s)\varepsilon_0}(x))}.$$

Recall that $G_U(x,\varepsilon,\varepsilon_0)$ is given by (1.2) in Assumption 3(b). Note that if ε_0 is small, then $\operatorname{supp}G_U(\cdot,\varepsilon,\varepsilon_0)$ and $\operatorname{supp}G_R(\cdot,\varepsilon,\varepsilon_0)$ are disjoint. Also, by (1.1) and Lemma 6.1, we know that

$$G(\varepsilon,\varepsilon_0)=\sup_{x\in M}\{G_U(x,\varepsilon,\varepsilon_0),G_R(x,\varepsilon,\varepsilon_0)\}\le\frac{\lambda\varepsilon^{\alpha}}{\varepsilon_0^{\alpha}}.\tag{6.9}$$

Then we take

$$D=2\zeta+2(1+\zeta\varepsilon^{\alpha})\sup_{\varepsilon\le\varepsilon_0}G(\varepsilon,\varepsilon_0)\varepsilon^{-\alpha}+K'b\gamma_m^{-1}.\tag{6.10}$$

By (6.9), $G(\varepsilon,\varepsilon_0)\varepsilon^{-\alpha}\le\lambda\varepsilon_0^{-\alpha}$. We have $D<\infty$.

Let $\varepsilon \le \varepsilon_0$.

By Proposition 5.1,

$$\operatorname{osc}(\hat{P}f,\ B_\varepsilon(x)) \le \sum_{j=1}^{K}\sum_{i=0}^{\infty}\operatorname{osc}\big((f\hat{g})\circ\hat{T}_{ij}^{-1}\chi_{\hat{T}U_{ij}},\ B_\varepsilon(x)\big)$$

$$\le \sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big(\operatorname{osc}\big((f\hat{g})\circ\hat{T}_{ij}^{-1}, B_\varepsilon(x)\big)\chi_{\hat{T}U_{ij}}(x) + \big[2\operatorname*{Esup}_{B_\varepsilon(x)}(f\hat{g})\circ\hat{T}_{ij}^{-1}\big]\chi_{B_\varepsilon(\partial\hat{T}U_{ij})}(x)\Big)$$

$$=:\sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big(R_{ij}^{(1)}(x)\chi_{\hat{T}U_{ij}}(x) + R_{ij}^{(2)}(x)\chi_{B_\varepsilon(\partial\hat{T}U_{ij})}(x)\Big). \tag{6.11}$$

Denote $y_{ij} = \hat{T}_{ij}^{-1}x$. We can choose $N = N(\varepsilon) > 0$ for each $0 < \varepsilon \le \varepsilon_0$ according to Assumption 4(b).

For $R_{ij}^{(1)}(x)$ with $x \in \hat{T}U_{ij}$, we first consider the case $i \le N(\varepsilon)$. By Assumption 4(a), (b) and (6.7), we have $\hat{g}(y'_{ij})/\hat{g}(y_{ij}) \le (1 + Js^\alpha\varepsilon^\alpha)(1 + cs^\alpha\varepsilon^\alpha) \le 1 + \zeta\varepsilon^\alpha$ if $d(T^{i+1}y_{ij}, T^{i+1}y'_{ij}) \le s\varepsilon$. Hence $\hat{g}(y'_{ij}) \le (1+\zeta\varepsilon^\alpha)\hat{g}(y_{ij})$ and $\operatorname{osc}(\hat{g},\ B_{s\varepsilon}(y_{ij})) \le 2\zeta\varepsilon^\alpha\hat{g}(y_{ij})$. So we get

$$\begin{aligned}
&R_{ij}^{(1)}(x) = \operatorname{osc}\big(f\hat{g},\ \hat{T}_{ij}^{-1}B_\varepsilon(x)\cap U_{ij}\big)\\
\le\ &\operatorname{osc}\big(f,\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\operatorname*{Esup}_{B_{s\varepsilon}(y_{ij})\cap U_{ij}}\hat{g} + \operatorname{osc}\big(\hat{g},\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\operatorname*{Einf}_{B_{s\varepsilon}(y_{ij})\cap U_{ij}} f\\
\le\ &(1+\zeta\varepsilon^\alpha)\operatorname{osc}\big(f,\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\hat{g}(y_{ij}) + 2\zeta\varepsilon^\alpha f(y_{ij})\hat{g}(y_{ij}).
\end{aligned}$$

If $i > N(\varepsilon)$, then we must have $x \in R_0$, and therefore

$$\begin{aligned}
&R_{ij}^{(1)}(x) = \operatorname{osc}\big(f\hat{g},\ \hat{T}_{ij}^{-1}B_\varepsilon(x)\cap U_{ij}\big)\\
\le\ &\operatorname{osc}\big(f,\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\operatorname*{Einf}_{B_{s\varepsilon}(x)\cap U_{ij}}\hat{g} + \operatorname{osc}\big(\hat{g},\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\operatorname*{Esup}_{\hat{T}_{ij}^{-1}B_\varepsilon(x)} f\\
\le\ &\operatorname{osc}\big(f,\ \ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\hat{g}(y_{ij}) + \|f\|_\infty \sup_{\hat{T}_{ij}^{-1}B_\varepsilon(x)}\hat{g}.
\end{aligned}$$

By Assumption 4(b), for any $x \in R_0$, $\displaystyle\sum_{i=N}^{\infty}\big(\sup_{\hat{T}_{ij}^{-1}B_\varepsilon(x)}\hat{g}\big) \le b\varepsilon^{m+\alpha}$. Hence,

$$\begin{aligned}
&\sum_{j=1}^{K}\sum_{i=0}^{\infty}R_{ij}^{(1)}(x)\chi_{\hat{T}U_{ij}}(x) \le K'b\varepsilon^{m+\alpha}\|f\|_\infty\chi_{R_0}(x)\\
+\ &\sum_{j=1}^{K}\sum_{i=0}^{\infty}\Big((1+\zeta\varepsilon^\alpha)\operatorname{osc}\big(f,\ B_{s\varepsilon}(y_{ij})\cap U_{ij}\big)\hat{g}(y_{ij}) + 2\zeta\varepsilon^\alpha f(y_{ij})\hat{g}(y_{ij})\Big)\\
\le\ &K'b\varepsilon^{m+\alpha}\|f\|_\infty\chi_{R_0}(x) + (1+\zeta\varepsilon^\alpha)\big[\hat{P}\operatorname{osc}\big(f,\ B_{s\varepsilon}(\cdot)\big)\big](x) + 2\zeta\varepsilon^\alpha(\hat{P}f)(x).
\end{aligned}$$

Since $\int_{\hat{M}} \hat{P} f d\hat{\nu} = \int_{\hat{M}} f d\hat{\nu}$ for any integrable function f, we have

$$\begin{aligned}
&\int_{\hat{M}} \sum_{j=1}^{K} \sum_{i=0}^{\infty} R_{ij}^{(1)} \chi_{\hat{T}U_{ij}} d\hat{\nu} \\
\leq \quad & K' b \varepsilon^{m+\alpha} \|f\|_\infty \hat{\nu} R_0 + (1+\zeta\varepsilon^\alpha) \int_{\hat{M}} \operatorname{osc}\big(f,\ B_{s\varepsilon}(\cdot)\big) d\hat{\nu} + 2\zeta\varepsilon^\alpha \int_{\hat{M}} f d\hat{\nu} \\
\leq \quad & (1+\zeta\varepsilon^\alpha) s^\alpha \varepsilon^\alpha |f|_\alpha + 2\zeta\varepsilon^\alpha \|f\|_1 + K' b \varepsilon^{m+\alpha} \|f\|_\infty \hat{\nu} R_0. \qquad (6.12)
\end{aligned}$$

As for $R_{ij}^{(2)}(x)$, if $i \leq N(\varepsilon)$, then we have

$$\operatorname*{Esup}_{B_\varepsilon(x)} (f\hat{g}) \circ \hat{T}_{ij}^{-1} \leq \big[\operatorname*{Esup}_{B_{s\varepsilon}(y_{ij})} |f| \big] \hat{g}(y_{ij})(1+\zeta\varepsilon^\alpha).$$

Hence by the same method as in [17], we get that

$$\int_{\hat{M}} \sum_{j=1}^{K} \sum_{i=0}^{N(\varepsilon)} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial\hat{T}U_{ij})} d\hat{\nu} \leq 2(1+\zeta\varepsilon^\alpha) G(\varepsilon,\varepsilon_0) \big(\varepsilon_0^\alpha |f|_\alpha + \|f\|_1\big).$$

If $i \geq N(\varepsilon)$, then $\operatorname*{Esup}_{B_\varepsilon(x)} (f\hat{g}) \circ \hat{T}_{ij}^{-1} \leq \|f\|_\infty \sup_{\hat{T}_{ij}^{-1} B_\varepsilon(x)} \hat{g}$, and

$$\sum_{j=1}^{K} \sum_{i=N(\varepsilon)}^{\infty} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial\hat{T}U_{ij})} \leq 2K' \|f\|_\infty \sum_{i=N(\varepsilon)}^{\infty} \sup_{\hat{T}_{ij}^{-1} B_\varepsilon(x)} \hat{g}$$

Again, by Assumption 4(b) it is bounded by $2K' b \varepsilon^{m+\alpha} \|f\|_\infty$. So we have

$$\begin{aligned}
&\int_{\hat{M}} \sum_{j=1}^{K} \sum_{i=0}^{\infty} R_{ij}^{(2)} \chi_{B_\varepsilon(\partial\hat{T}U_{ij})} d\hat{\nu} \\
\leq \ & 2(1+\zeta\varepsilon^\alpha) G(\varepsilon,\varepsilon_0) \big(\varepsilon_0^\alpha |f|_\alpha + \|f\|_1\big) + 2K' b \varepsilon^{m+\alpha} \|f\|_\infty \hat{\nu} B_\varepsilon(\partial R_0). \qquad (6.13)
\end{aligned}$$

We may assume that $\hat{\nu} R_0 + \hat{\nu} B_\varepsilon(\partial R_0) \leq 1$. By Proposition 5.2(i) and (5.3) we have that $\varepsilon^{m+\alpha} \|f\|_\infty \leq \gamma_m^{-1} \varepsilon^\alpha \|f\|_\alpha$ and $\|f\|_\alpha = |f|_\alpha + \|f\|_1$ respectively. So by (6.11), (6.12) and (6.13), we get

$$\begin{aligned}
\int_{\hat{M}} \operatorname{osc}\big(Pf,\ B_\varepsilon(\cdot)\big) d\hat{\nu} \ \leq \ & \big[(1+\zeta\varepsilon^\alpha)\big(s^\alpha\varepsilon^\alpha + 2G(\varepsilon,\varepsilon_0)\varepsilon_0^\alpha\big) + 3K' b \gamma_m^{-1} \varepsilon^\alpha\big] |f|_\alpha \\
+ \ & \big[2\zeta\varepsilon^\alpha + 2(1+\zeta\varepsilon^\alpha) G(\varepsilon,\varepsilon_0) + 3K' b \gamma_m^{-1} \varepsilon^\alpha\big] \|f\|_1.
\end{aligned}$$

Now the result follows by the choice of η and D in (6.8) and (6.10). $\square$

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